The Drèze and Grossman-Hart criteria for production in incomplete markets: Voting foundations and compared political stability*

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Abstract

This paper studies corporate control in a general equilibrium model with incomplete markets. At the market equilibrium, shareholders typically disagree on the way to evaluate production plans outside the market span. Hence a collective decision mechanism is needed to resolve this conflict. A mechanism proposed by Drèze (1974), resp. Grossman & Hart (1979), consists in allowing (Lindahl-like) sidepayments between final, resp. initial, shareholders. Although it is likely to exhibit desirable efficiency properties, this mechanism is difficult to implement. Another mechanism (e.g., Drèze (1985) and De-Marzo (1993)) relies on majority voting by shareholders. Since voting occurs in a multi-dimensional setup, super majority rules are needed to ensure existence of equilibria. We give conditions under which sidepayment equilibria are voting equilibria for the smallest rate of super majority ensuring existence. Thereby we are able to compare the relative performances of the Drèze and Grossman-Hart criteria with respect to stability in the voting mechanism. We show that the endogenization of portfolio holdings in the Drèze criterion can either help or completely jeopardize the aggregation process, depending on the shareholders’ expectations. This ambivalence is absent in the Grossman-Hart criteria.

Keywords: Incomplete markets, super majority voting, sidepayments, corporate charter, self-fulfilling prophecies.

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1 Introduction

In general equilibrium models of production economies with incomplete markets, agents (consumers/shareholders) trade assets but, at the market equilibrium, they typically disagree on the way to evaluate income streams outside the market span. Indeed, in any equilibrium, investors price assets by using their probability and time-preference adjusted marginal rates of substitution (called ‘shadow price vectors’ in the sequel). If the financial markets are complete, these individual shadow prices are all equal and correspond to the Arrow-Debreu state prices. If the markets are incomplete, then, in general, shadow prices are not equal across consumers. Hence profit maximization is not a well defined objective for the firm\(^1\).

Among the ways that have been proposed in the literature to resolve these conflicts, two are compared in the present study. The first way, proposed by Drèze (1974) and Grossman & Hart (1979), consists in allowing sidepayment between shareholders. At equilibrium, there is no alternative investment policy that makes everybody better off, even if one allows for sidepayments between shareholders\(^2\). Drèze (1974) argues through efficiency while Grossman & Hart (1979) argue through competitive behavior. But in both cases, similar criteria are proposed as an objective for the firm: profit should be maximized with respect to a system of shadow prices that averages the idiosyncratic shadow prices of all shareholders. The difference is that Drèze (1974) uses the final share holdings to average the shadow prices, whereas Grossman & Hart (1979) uses the initial share holdings.

A second way to resolve these disputes between shareholders is based on majority voting in assemblies of shareholders\(^3\). Among others Drèze (1985) and DeMarzo (1993) propose the same concept of stability for production equilibria: at equilibrium, within each firm, the production plans of other firms remaining fixed, no alternative production plan can rally a majority of shareholders/shares (depending on whether the governance is of the ‘one person—one vote’ or ‘one share—one vote’ type) against the status quo. As Gevers (1974) already noted, the first problem this approach runs into is existence: Plott (1967) shows that in multi-dimensional voting models, a simple majority political equilibrium

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\(^1\)For details on standard general equilibrium models of production with incomplete markets and the role of the firms, see, e.g., Magill & Quinzii (1996) and the references therein as well as the more recent study Dierker, Dierker & Grodal (2002).

\(^2\)Grossman & Hart (1979) describes a takeover scheme where a ‘new’ manager, proposing an alternative investment policy, solicits payments from the shareholders who perceive they derive benefits from the change to buy proxies from the shareholders who perceive that the change impairs their utility levels.

\(^3\)The choice of a state contingent production plan in a publicly traded corporation is a genuine problem of social choice. This problem has been profoundly important in the history of economic thought since Arrow’s impossibility theorem arose out of his effort to find mechanisms for solving disagreements in such cases.
typically does not exist\textsuperscript{4}. A way to restore existence is to install a super majority rule\textsuperscript{5}: to overturn the status quo, a challenger should rally a proportion bigger than the simple majority of the voting population. The question then arises of a ‘suitable’ level of a super majority, $\rho \in [1/2, 1]$: it should be high enough to secure existence, and low enough not to be too conservative. The standard way to proceed is to associate to each proposal its (Simpson-Kramer) score. The score of a proposal (the incumbent, or status quo) is the fraction of the voting population supporting, against this proposal, its most dangerous challenger, i.e., the alternative proposal that rallies the maximal fraction of voters against the incumbent. The most stable proposals are the ones with lowest score (the so-called min-max).

The object of the present paper is to study the extent to which these two collective decision mechanisms (sidepayments and super majority voting) are equivalent. Starting from an initial production plan, investors value any new plan using their shadow prices. If the shadow prices are not equal, then investors could either vote using their private valuations of the new plan, or those investors better off could buy off other investors to reach unanimity. But there is no clear intuition that these two processes select the same production plans as ‘political equilibria’. There is nevertheless a clear formal link between the two approaches: on the one hand, sidepayments equilibria satisfy the Drèze/Grossman-Hart criteria which states that profit should be maximized with respect to the mean shadow price vectors of all shareholders; on the other hand we know from Caplin & Nalebuff (1988, 1991) that the ‘mean voter’ is likely to be stable with respect to the smallest possible rate of super majority (and is a good proxy of the Simpson-Kramer min-max).

It is therefore natural to study under which conditions sidepayment equilibria are ‘legitimate’ from a voting perspective. It happens to be that, under certain conditions on the form and distribution of primitive characteristics of the agents, the Drèze/Grossman-Hart equilibria based on sidepayments are stable for the smallest rate of super majority ensuring existence, a rate smaller than 64%. Hence, under these conditions, the majority voting mechanism is likely to implement the sidepayment equilibria. We believe that this result is important for the following reasons: Sidepayments equilibria are attractive since they fulfill the first order conditions of constrained Pareto efficiency (see Magill & Quinzii (1996)). But they are based on a (Lindahl-like) process of exchange through ‘personalized’ prices, since shareholders are supposed to contribute to a proposed change in the produc-

\textsuperscript{4}If production possibility frontiers are unidimensional, then Benninga & Muller (1979) have shown that a simple majority voting scheme always works in the present context. In the present paper, production possibility frontiers are multi-dimensional, therefore the need for a super majority voting rule or sidepayments to restore existence of political equilibria. It is worth noticing that another condition ensuring existence of simple majority political equilibria is that the degree of market incompleteness is equal to one, see Tvede & Crèes (2001).

\textsuperscript{5}To get existence, Drèze (1985) gives veto power to some shareholders. This result is generalized in Kelsey & Milne (1996) to encompass other voting rules, such as the generalized median voter rules, a special case of which has been applied to decision theory in firms by Sadanand & Williamson (1991).
tion plan according to her/his marginal utility. Hence sidepayments (which are often illegal) are generally considered difficult to implement. So from a positive point of view it is attractive to study equilibria in which decisions are made by (super) majority voting without sidepayments. And we believe that stability of production choices with respect to super majority voting is a good positive justification for the Drèze/Grossman-Hart criteria compared to stability with respect to unanimity in the presence of sidepayments.

It will be argued in the paper that the Drèze criterion satisfies super majority stability more easily (i.e., under weaker conditions) than the Grossman-Hart criterion because of the endogeneity of the shareholdings (Drèze uses a weighted-average based on final holdings whereas Grossman-Hart use a weighted-average based on initial holdings). The intuition is that of a ‘clientele effect’ in which investors with similar insurance needs take positions in the same firms, thus aligning their interests within a firm, a self-selection process which endogenously homogenizes the individual characteristics of the social choice problem and makes it easier to solve\(^6\). A concrete technical support of this intuition is provided at various places in the paper.

Since the electorate is endogenous, its composition is influenced by the agents’ expectations. The classical concept of majority voting equilibrium supposes that shareholders have ‘conservative’ expectations: they expect that no challenger can defeat the status quo; therefore they believe that the status quo prices are going to prevail in the future; so they stick to their current shares. In equilibrium, given these current shares, conservative expectations are self-fulfilling: no challenger can rally a high enough majority against the status quo. Hence voting equilibria may be viewed as plain Nash equilibria (see Drèze (1989), pp. 48-49).

But what happens if shareholders deviate from these conservative expectations? If they expect a challenger to defeat the status quo, they anticipate new equilibrium prices and compute new equilibrium shares; for these expectations to be fulfilled, it must be the case that, given the new distribution of voting weights/shares, the challenger rallies a high enough majority against the status quo.

Intuitively, an equilibrium stable with respect to a super majority of rate \(\rho\) under conservative expectations need not be stable if agents’ expectations deviate. Indeed, the shareholders whose insurance needs are less met by the investment policy of the challenger than by the incumbent’s will exit, at least partially, from the capital of the firm. They will sell (part of) their shares to the shareholders whose insurance needs are better met by the challenger’s investment policy than by the incumbent’s. The exit effect gives more voting weight, in the corporate control mechanism, to the shareholders who favor the challenger over the status quo. This alternative expectation regime, dubbed ‘exit\(^7\), always gives an

\(^6\)Tvede & Crès (2001) reveals that trading shares reduces the dimension of conflict in the social choice problem because it homogenizes the shareholders’ preferences on the market span. The present paper shows that endogenizing voting weights through trades on shares also helps solving the social choice problem.

\(^7\)The choice of the term exit is a tribute we pay to the path-breaking work of Hirschman (1970).
advantage to the challenger in the voting process.

It is shown that generically a (weakly) higher rate than $\rho$ is necessary for the corporate charter to secure that a $\rho$-majority equilibrium is immune to deviation of expectations. Robust examples, where a strictly higher rate is needed to ensure exit-stability, are provided. The extent to which the corporate charter needs to be increased to secure that a $\rho$-majority equilibrium is exit-stable depends on the case under consideration. But aggregation of preferences can be completely jeopardized. Indeed, we give an example where a unique 50%-majority equilibrium under conservative expectations exists which will not be exit-stable for any rate of super majority except unanimity, and no other production plan can do better. This example gives rise to Condorcet cycles between two alternatives, even for rates of super majority as close to unanimity as one wants, a property due to self-fulfilling prophecies through exchange on markets. The endogenization of the electorate through trade is thus double-edged. Of course, the Grossman-Hart criterion is immune to such political self-fulfilling prophecies.

The paper is constructed as follows. Section 2 sets up the model and states the main result: under some conditions on the distribution of endogenous variables, sidepayment equilibria happen to have nice stability properties with respect to majority voting. Section 3 introduces a sharp and crisp leading example where the distribution of endogenous variables can easily be computed and linked to assumptions on the set of primitive characteristics; moreover, a concrete technical support of the intuition underlying the clientele effect is provided and discussed. Section 4 focuses on deviations from the so-called conservative expectations underlying the traditional voting equilibria; it introduces a strong version of the stability criterion with respect to majority voting: the exit-stability; finally, through a simple geometric example, it explores the possible occurrence of political self-fulfilling prophecies, and of Condorcet cycles of length two for any rate of super majority. Finally, Section 5 proposes an extension of the assumptions under which the main argument holds true.

2 The Model and central result

2.1 The setup

Consider a finance economy with two periods, $t \in \{0, 1\}$, and $S$ states of nature in period 1, indexed by $s$, $s \in \{1, \ldots, S\}$ ($s = 0$ denotes the initial period). There is one good in each state. There is a finite measure space of agents $(I, P, \mu)$, and a measurable mapping from this space into the set of agents primitive characteristics. A generic agent, denoted $i$, Hirschman was concerned with the attitude of consumers confronted to a decline in quality of a product and opposed (i) consumers/shareholders who choose to exit when they are not happy about the product/policy to (ii) the loyal consumers who prefer to stay in and wait for better days. Our model is more a model of horizontal, rather than vertical, differentiation. Nevertheless, Hirschman’s opposition is present in the context of this paper.
is endowed with a vector of initial endowments \( \omega_i = (\omega_i^0, \omega_i^1, \ldots, \omega_i^S) \in \mathbb{R}^{S+1} \) and a utility function \( u_i \) from \( \mathbb{R}^{S+1} \) to \( \mathbb{R} \), which is assumed to be continuous, differentiable, strictly increasing and strictly quasi-concave.

There are \( J \) firms indexed by \( j, j \in \{1, \ldots, J\} \). Firm \( j \) is characterized by its production set \( \mathcal{Y}_j \subset \mathbb{R}^{S+1} \). These production sets are assumed to be convex and smooth, and moreover the set of efficient production plans \( \mathcal{Z}_j \) (defined by \( \mathcal{Z}_j = \{y_j \in \mathcal{Y}_j | \{y_j\} + \mathbb{R}^{S+1} \cap \mathcal{Y}_j = \{y_j\}\} \)) are bounded. Firms are owned by agents. The initial distribution of shares within firm \( j \) is described by a real function \( \delta_{i,j} \) over \( \mathcal{I} \), satisfying
\[
\int_{\mathcal{I}} \delta_{i,j} \ d\mu(i) = 1, \text{ for all } j.
\]

### 2.2 Stock market equilibria with fixed production plans

Given announced production plans, \( y = (y_j)_{j=1}^{J} \in \prod_{j=1}^{J} \mathcal{Z}_j \) (it is assumed that firms only announce efficient production plans, and that the \( y_j \)'s are taken in general position), a market span \( \langle Y \rangle \), of dimension \( J \), is available for agents to trade in, where \( Y \) denotes the payoffs matrix:
\[
Y = \begin{pmatrix}
y_1^1 & \cdots & y_1^J \\
\vdots & \ddots & \vdots \\
y_S^1 & \cdots & y_S^J
\end{pmatrix}.
\]

Denote \( y^* \in \mathbb{R}^J \) the \( s \)-th row of the matrix \( Y \), \( s \in \{1, \ldots, S\} \), and \( y^0 \) the vector of period 0 inputs. Agents trade the \( J \) equity contracts \( (y_j)_{j=1}^{J} \) at market prices, \( q = (q_j)_{j=1}^{J} \in \mathbb{R}^J \).

The budget set of a generic agent \( i \) is:
\[
B_i(y, q) = \left\{ x_i \in \mathbb{R}^{S+1} | \exists \theta_i \in \mathbb{R}^J : \begin{align*}
x_i^0 &= \omega_i^0 + q \cdot \delta_i + \delta_i \cdot y^0 \\
x_i^s &= \omega_i^s + \theta_i \cdot y^s \quad \text{for all } s \neq 0
\end{align*} \right\}.
\]

**Definition 1** \((x^*, \theta^*, q^*)\) is a stock market equilibrium with fixed production plans \( y \) (SME\((y)\)) if

- **agent optimize:** for all \( i \in \mathcal{I} \),
  \[
  x_i^* = \arg \max \{u_i(x) | x \in B_i(y, q^*)\}
  \]

- **markets clear:** for all \( j \in \{1, \ldots, J\} \)
  \[
  \int_{\mathcal{I}} \theta_{i,j}^* \ d\mu(i) = 1.
  \]

At a SME\((y)\), for every agent \( i \) one can define the **normalized** equilibrium gradient: \( \nabla_i^* \), where the first coordinate is normalized to 1. This gradient is the shadow price vector of agent \( i \) at equilibrium and it consists of the marginal rates of substitution of agent \( i \) for wealth between period 0 and states of nature at period 1. In models with finitely many
consumers, generically with respect to endowments, at equilibrium these shadow price vectors are different two by two: there is no pair of shareholders with a common view on the way to value income streams outside the market span (see, e.g., Magill & Quinzii (1996) Theorem 11.6).

Of course, the way production plans are chosen should be endogenized. It is clear that production plans should be ‘best’ with respect to some objective for the firm. The objectives we are interested in are those based on some collective decision mechanism. Hence the full-fledged equilibrium concepts we aim at are equilibrium situations with respect to the simultaneous operation of both an (decentralized) exchange mechanism and a (centralized) collective decision mechanism. Among the latter, first we focus on the sidepayment mechanism and second on the super majority voting mechanism. This is the object of the next two subsections.

2.3 Stock market equilibria

Drèze (1974) and Grossman & Hart (1979) define the concept of stock market equilibrium based on allowing sidepayments between disagreeing shareholders. Within each firm, an alternative production plan to the status quo can be proposed to shareholders; the fundamental scheme is that of a ‘takeover’ where the ‘new’ manager, proposing the alternative investment policy, solicits payments (in period 0) from the shareholders who (perceive they) derive benefits from the change to buy proxies from the shareholders who (perceive they) derive losses from the change. At equilibrium no alternative production plan and set of transfers between shareholders, such that all shareholders are better off, exists. It is shown that, at equilibrium, the production plans chosen by firms are those which maximize profits with respect to the mean shadow price vector of the shareholders\(^8\). The weights to average shadow price vectors are the shares of the final shareholders for Drèze (1974) and initial shareholders for Grossman & Hart (1979).

In the present setup this naturally leads to the definition of stock market equilibria where the production plans chosen by firms are those that maximize profits with respect to the mean shadow prices of the shareholders for a given ownership structure. The latter concept consists of a map \(\eta : \mathcal{I} \to \mathbb{R}_+^J\) whose \(j\)’s component \(\eta_i\) is a density on \(\mathcal{I}\) characterizing the ownership structure within firm \(j\). An intuitive interpretation of the function is that \(\eta_{i,j}\) is the ‘weight’ given to agent \(i\) in the selection process of the production plan. Thus, it characterizes a mode of governance. Hence this last definition corresponds to the Drèze criterion for \(\eta \equiv \theta^+\), and to the Grossman-Hart criterion for \(\eta \equiv \delta^+\).

\(^8\) This equilibrium concept is a refinement of the Pareto criterion. A stock market equilibrium with fixed production plans, \(\mathcal{E} = (y, q, x, \theta)\), is supposed to satisfy the Pareto criterion if, within each firm, there does not exist an alternative production plan which makes all shareholders better off —without sidepayment. A necessary and sufficient condition is that profits be maximized with respect to some price vector in the convex hull of all shareholders’ shadow price vectors. See Magill & Quinzii (1996).
At any SME \((y)\) one can define the induced support \(\mathcal{N}_E \in \mathbb{R}^{S+1}\) of normalized equilibrium gradients, \(\nabla^*_E\). Due to the first order condition of the agents’ optimization problem, \(\mathcal{N}\) is included in a subspace of dimension \(S - J\) of \(\mathbb{R}^{S+1}\), a property which has some interesting consequences in terms of social choice (see the discussion in Section 3.2 below). Moreover, the original probability distribution \(\mu\) together with the ownership structure \(\eta\) induce a distribution \(\pi\) on \(\mathcal{N}\).

**Definition 2** Given an ownership structure \(\eta\), at a stock market equilibrium with fixed production plans, \(E = (x^*, \theta^*, q^*, y)\), let the mean shadow price vector for firm \(j\) be \(\nabla_j(\eta) \in \mathbb{R}_{++}^{S+1}\) defined by

\[
\nabla_j(\eta) = \int_{\mathcal{N}} \nabla d\eta_j(\nabla).
\]

Then a stock market equilibrium \(E = (x^*, \theta^*, q^*, y^*)\) for the ownership structure \(\eta\), denoted SME(\(\eta\)) in the sequel, is a stock market equilibrium \((x^*, \theta^*, q^*)\) with fixed production plans \(y^*\), such that for all \(j\):

\[
\forall j : y^*_j \in \arg\max\{\nabla_j(\eta) \cdot z_j \mid z_j \in Z_j\}. \tag{1}
\]

Though our focus is not existence of stock market equilibria, it should be noted that existence has been established for the Grossman-Hart criterion in Magill & Quinzii (1988) and, in case short sales are prohibited, for the Drèze criterion in Geanakoplos, Magill, Quinzii & Drèze (1990). In Momi (2001) a robust example of non-existence for the Drèze criterion without short sales constraints is provided. We do not prohibit short sales, but our results and examples extend to that case with the obvious modifications. In our examples either no shareholder goes short or additional assumptions could ensure that no shareholder goes short.

### 2.4 Majority stable equilibria

We turn now to the classical concept of majority stable equilibrium (see, e.g., Drèze (1985, 1989), DeMarzo (1993), Kelsey & Milne (1996)). We focus on the stock market equilibria with fixed production plans where, at the equilibrium allocations of goods and shares, production plans cannot be defeated by any challenger, given the governance. Informally, a SME(\(y\)) is \(\rho\)-majority stable provided that, within each firm, production plans of other firms remaining fixed, there is no alternative production plan preferred by more than \(\rho \times 100\) percent of the owners to the status quo. A more formal description follows.

Suppose, at a SME(\(y\)) \((x, \theta, q)\), that shareholders have *conservative* expectations in the sense that they do not expect any challenger to be able to win a proxy fight against the status quo in any of the \(J\) firms. Then all shareholders expect the prices \(q\) to prevail in the future, so they stick to their optimal portfolios \(\theta\). When given the choice between a challenger, \(y'_j \in Y_j\), and the status quo, \(y_j\), all other production plans \(y_{-j} = (y_k)_{k \neq j}\) being fixed, shareholder \(i\) prefers the challenger if and only if

\[
u_i(x_i + \theta_{i,j}(y'_j - y_j)) > u_i(x_i).
\]
Let \( I(x, \theta, y_i', y_j') \subset I \) be the set of agents satisfying the latter inequality. If expectations are confirmed at equilibrium, then it should be the case that the relative weight, in the voting process, of the coalition \( I(x, \theta, y_j, y_I) \) is smaller than the rate \( \rho \) of the corporate charter.

The formal description of stability with respect to voting goes through the construction of preferences for firms over production plans for a given ownership structure \( \eta \). These preferences are described by correspondences, \( P^{\rho}_j \):

\[
P^{\rho}_j(x, \theta, y_i) = \begin{cases} 
\emptyset & \text{for } \int_I \eta_{i,j} d\mu = 0 \\
\left\{ y_i' \in \mathcal{Y}_i \mid \int_I(x, \theta, y_i, y_i') \frac{\int_I \eta_{i,j} d\mu}{\int_I \eta_{i,j} d\mu} > \rho \right\} & \text{for } \int_I \eta_{i,j} d\mu > 0.
\end{cases}
\]

Of course, the status quo is stable with respect to majority voting if no firm has an alternative policy in its preferred set. At equilibrium, there is stability simultaneously with respect to exchange and voting.

**Definition 3** \((x^*, \theta^*, q^*, y^*)\) is a \( \rho \)-majority stable equilibrium \((\rho\text{-MSE}(\eta))\) if:

- \((x^*, \theta^*, q^*)\) is a stock market equilibrium with fixed production plans \( y^* \),
- no challenger defeats the status quo: for all \( j \), \( P^{\rho}_j(x^*, \theta^*, y_j^*) = \emptyset \).

In fact, it follows immediately from the quasi-concavity of the utility functions that the most dangerous challengers for the status quo are infinitesimally close to it. Hence the optimal strategy for the challenger is always minimal differentiation.

**Lemma 1** At a SME\((y) (x, \theta, q)\), for all \( j \) and all open balls \( B \) centered et \( y_i \) in \( \mathbb{R}^{S+1} \),

\[
P^\rho_j(x, \theta, y_j) \cap B = \emptyset \implies P^\rho_j(x, \theta, y_j) = \emptyset.
\]

Thanks to this property one can restrict attention to infinitesimal changes, when considering alternative proposals to the status quo. For any \( y_i \in \mathcal{Z}_j \) there exists a supporting price \( p_i \in \mathbb{R}^{S+1}_+ \setminus \{0\} \) such that \( y_i \) maximizes profits for firm \( j \) with respect to \( p_i \). Suppose that shareholders propose an infinitesimal change \( \varepsilon t_j \), where \( t_j \in \langle p_i \rangle_\perp \) and \( \varepsilon > 0 \), of the production plan \( y_j \) (where \( \langle p_i \rangle_\perp \) is the hyperplane orthogonal to \( p_i \)); agent \( i \) supports the change (for \( \varepsilon \) sufficiently small) if and only if:

\[
\theta_{i,j} \nabla_i \cdot t_j > 0.
\]  \hspace{1cm} (2)

This naturally leads us to conduct our study of majority-stable equilibria in the space of gradients, where the geometry of the voting problem is transparent.
2.5 The central result

Let us recall that a function, $g$, is $\sigma$-concave over a convex set $A$ if for all $a, a' \in A$ and all $t \in [0, 1]$, $g((1-t)a + ta') \geq ((1-t)g(a) + tg(a'))^{1/\sigma}$.

This assumption is regarded in the literature (see Caplin & Nalebuff (1988, 1991)) as imposing some degree of homogeneity on the considered space of characteristics. (Note that $r = \infty$ yields a constant function.) Define the rate of super majority:

$$r(\alpha) = 1 - \left( \frac{S - J + \alpha}{S - J + 1 + \alpha} \right)^{S-J+\alpha}.$$  

The central result of the paper follows.

Proposition 1 Consider a SME($\eta$), $E$. If the induced probability distribution $\eta$ on $N$ has compact and convex support and has a $\sigma$-concave density for some $\sigma > 0$, then $E$ is a $r(1/\sigma)$-majority stable equilibrium; hence 0.64-majority stable.

Proof: At a SME($\eta$), $E$, any proposed direction of infinitesimal change $t_j$ is, by construction, orthogonal to the price vector with respect to which profit is maximized, i.e., $\nabla_j(\eta) \cdot t_j = 0$. Therefore inequality (2) delimits a half space cutting the set of normalized equilibrium gradients (by the hyperplane $\langle t_j \rangle^\perp$) in two pieces through its centroid. The proposition is a direct application of Theorem 1 in Caplin & Nalebuff (1991).\hfill $\square$

The next sections link the conditions under which the proposition rests to assumptions on the primitive characteristics of the model.

It will be demonstrated that the proposition holds under weaker conditions for a governance à la Drèze than for a governance à la Grossman-Hart: the endogenization of share holdings in the former governance helps solving the social choice problem. This discussion will be led at first on a special case where it is particularly transparent.

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9 The ratio $r(\alpha)$ is increasing in $\alpha$ and bounded from above by $1 - 1/e \approx 0.632$ when $\alpha \geq 2 - S + J$.

10 In loose terms, the theorem states that there is no way to cut, by a hyperplane, a compact and convex support endowed with a $\sigma$-concave distribution through its centroid in such a way that one of the two resulting pieces contains more than $100 \ r(1/\sigma)$ percent of the weight. The game played is simple: two players have to share a ‘cake’—the support $A$ endowed with a $\sigma$-concave distribution $g$—; the first player indicates a point in the cake, the second one cuts the cake by a hyperplane, through the indicated point, as unevenly as possible and takes the biggest piece; what is left to the first player is the Simpson-Kramer score of the point he indicated. A point with high score (and good proxy of the min-max) is the centroid of the cake; its score is $1 - r(1/\sigma)$. 

10
The leading example: linear quadratic preferences

A setup in which the distribution of endogenous variables can easily be computed and linked to that of primitive characteristics is the case where agents have (CAPM-like\textsuperscript{11}) linear-quadratic utility functions:

\[
u_i(x) = \gamma^0 x^0_i + \sum_{s=1}^{S} \pi^s \left( \gamma^1 x^s_i - \frac{1}{2} (x^s_i)^2 \right),\]

where \(\pi = (\pi^1, \ldots, \pi^S)\) is the (common) vector of objective probabilities over states of nature at date 1; and \((\gamma^0, \gamma^1) \in \mathbb{R}^2\) are idiosyncratic characteristics of agent \(i\)'s preferences. Utility functions are thus of the mean-variance type.

A generic agent \(i\) is consequently indexed by a vector of primitive characteristics \(i = (\omega, \delta, \gamma) \in \mathbb{R}^{S+1} \times \mathbb{R}^J \times \mathbb{R}_+^2\). Define \((\Omega, \Gamma) \in \mathbb{R}^{S+3}\), the vector of aggregate characteristics, i.e., such that

\[\forall s: \int_{\mathcal{I}} \omega^s d\mu(\iota) = \Omega^s; \quad \int_{\mathcal{I}} \gamma^t d\mu(\iota) = \Gamma^t \text{ for } t = 0, 1.\]

It is assumed that \(\gamma^0 > 0\) and \(\gamma^1 > \max\{\Omega^1, \ldots, \Omega^S\}\) for all \(i \in \mathcal{I}\) in order to have monotone preferences on the relevant domain.

3.1 Properties of equilibria

Fix the following notation: the subscript \(1\) denotes the last \(S\) coordinates (corresponding to period one) of a vector. Let \(\Pi\) be the \(S\)-dimensional diagonal matrix, \(\text{Diag}(\pi_1, \ldots, \pi_S)\).

**Proposition 2** At a stock market equilibrium with fixed production plans,

\[
x_{i,1} = \omega_1 + YB(\gamma^1 1_S - \omega_1 - \gamma^0 \xi) \quad (4) \\
\theta_i = B(\gamma^1 1_S - \omega_1 - \gamma^0 \xi) \quad (5) \\
Du_i = \begin{pmatrix} \gamma^0 \\
\Pi YB \gamma^0 \xi + \Pi (I - YB)(\gamma^1 1_S - \omega_1) \end{pmatrix} \quad (6) \\
q = \xi^\top \Pi Y \quad (7)
\]

where \(B = (Y^\top \Pi Y)^{-1} Y^\top \Pi\) and \(\xi = \frac{1}{\Gamma^0}(\Gamma^1 1_S - \Omega_1 - Y^1 1_J)^{12}\)

\textsuperscript{11}Of course, the properties of the CAPM do not hold here because initial endowments are not supposed to lie in the span of assets. Thus agents disagree due to their different insurance needs outside the market span.

\textsuperscript{12}The matrix \(YB\) is associated with the \(\pi\)-projection on the market span \(\langle Y \rangle\).
Proof: Let us to prove the various expressions (see Magill & Quinzii (1996), section 17). The first order conditions of the optimization program of agent \( i = (\omega, \delta, \gamma) \) (orthogonality of the gradient to the matrix of wealth transfers) gives:

\[
Du_i^t \begin{pmatrix} \omega \\ Y \end{pmatrix} = 0 \iff \left( \Pi(\gamma^11_S - x_1) \right)^t \begin{pmatrix} -q \\ Y \end{pmatrix} = 0
\]

which, replacing \( x_1 \) by \( \omega_1 + Y\theta_i \), yields:

\[
\gamma^0q = (\gamma^11_S - \omega_1 - Y\theta_i)^t\Pi Y
\]

Then, integrating over \( \mathcal{I} \), market clearing gives equation (7):

\[
\Gamma_0q = (\Gamma^11_S - \Omega_1 - Y1_j)^t\Pi Y
\]

Then replacing \( q \) by its expression in (8) and transposing gives:

\[
Y^t\Pi(\gamma^11_S - \omega_1 - Y\theta_i) = Y^t\Pi\gamma^0\xi \iff Y^t\Pi Y\theta_i = Y^t\Pi(\gamma^11_S - \omega_1 - \gamma^0\xi)
\]

and \( Y^t\Pi Y \) being invertible when \( Y \) is full-ranked, one immediately gets, multiplying from the left by the inverse, equation (5). Then equation (4) obtains thanks to the budget constraints. Then equation (6) comes immediately. □

A noticeable aspect of the proposition is that endogenous variables do not depend on the initial distribution of shares. This comes from the linearity of the utility functions (with respect to period zero), which eliminates wealth effects between periods. However, this feature makes computations tractable and the discussion of the intuition more transparent.

A wide class of governances is considered in this section. They are of the form:

\[
\eta_{i,j} = \begin{cases} (\theta_{i,j})^a & \text{for } \theta_{i,j} > 0 \\ 0 & \text{for } \theta_{i,j} \leq 0 \end{cases}
\]

or

\[
\eta_{i,j} = \begin{cases} (\delta_{i,j})^a & \text{for } \delta_{i,j} > 0 \\ 0 & \text{for } \delta_{i,j} \leq 0 \end{cases}
\]

where \( a \) is a real number, depending on whether it is based on post-trade or pre-trade shares. Hence the classical ‘one shareholder—one vote’, resp. ‘one share—one vote’, governances are obtained for \( a = 0 \), resp. \( a = 1 \). Governances of the first, resp. second, form are denoted \( \theta^{+a} \), resp. \( \delta^{+a} \).

**Corollary 1** For the Dr`eze criterion: Suppose that the marginal distribution of \( \mu \) on \((\omega_1, \gamma)\) has compact, convex support and has a \( \sigma \)-concave density \( f \). Then for \( \eta \equiv \theta^{+a} \) a SME(\( \eta \)) is \( \rho \)-majority stable provided that \( \rho \geq r(S + 2 + a + 1/\sigma) \).

For the Grossman-Hart criterion: Suppose moreover that for all \( j \) the marginal distribution of \( \delta_{i,j}^{+} \) (taken as a measure on \( \mathcal{I} \)) on \((\omega_1, \gamma)\) has compact, convex support and has a \( \tau \)-concave density. Then for \( \eta \equiv \delta^{+a} \) a SME(\( \eta \)) is a \( \rho \)-majority stable provided that \( \rho \geq r(S + 2 + a/\tau + 1/\sigma) \).
Proof: Let us prove the first assertion (the second follows from the same line of argument). The portfolio mapping \( \theta_j \) is a linear function of \((\omega_1, \gamma)\). Hence the operator \( \theta_j^+ \) has a compact, convex support (it is the truncation by a hyperplane of the compact, convex support of the marginal distribution of \( \mu \) on \((\omega_1, \gamma)\)); moreover it is obviously 1/\(a\)-concave. The aggregate distribution of voting weights in firm \( j \) is the product of \( f \) and \( \theta_j^+ \) which is \( \sigma/(1 + a\sigma)\)-concave on the latter compact, convex support. (Indeed, Lemma 2 in the appendix shows that the product of a \( \sigma \)-concave and a \( \tau \)-concave distribution is \( \sigma\tau/(\sigma+\tau)\)-concave.) Therefore, the density, \( \eta_j \), induced by \( f\theta_j^+ \) on \( N \) is \( \sigma/[1+(S+2+a)/\sigma] \)-concave according to the Prékopa-Borell theorem (see proof of Proposition 4 below for details about that theorem and its use in the present context). □

An important property of this leading example is that the equilibrium distribution of shares is linear in primitive characteristics (thanks to the quadratic form of the utility functions). This linearity property, allied with the \( \sigma \)-concavity of the density \( f \) on endowment shocks and preference characteristics, guarantees the right concavity properties of the density of voting weights, \( f\theta_j^+ \) on \( I \). It is worth noticing that for the result to hold under a governance a la Grossman-Hart one needs additional assumptions on the distribution of initial shares. Hence our first observation: under the conservative expectations regime (nobody expects any challenger to be able to win a proxy fight against the status quo), trade helps voting to aggregate individual preferences: \( \sigma \)-concavity of the initial distribution of shares is needed under the Grossman-Hart criterion, while no assumption is needed under the Drèze criterion. In the next subsection we analyze a bit more in depth the intuition this finding.

### 3.2 Market-driven aggregation of individual preferences

The results obtained give hints about one of themes of the present paper: the market-driven endogenous structure on voting weights reduces the difficulty of aggregating individual preferences through majority voting.

As noted by Gevers (1974) and DeMarzo (1993), the main problem with the concept of majority equilibrium is that of existence: a simple (50 %) majority equilibrium typically does not exist. Existence is recovered here through a super majority rule. The central issue is to ‘tune’ for the right level of super majority: it should be high enough to secure existence, and low enough not to be too conservative. The ‘right’ level of super majority is the minmax.

The minmax is in general difficult to target exactly though. In the present setup, normalized gradients lie in a \((S-J)\)-dimensional space. Following the example of Greenberg (1979), Tvede & Crès (2001) provide an economy where the minmax is \( \frac{S-J+1}{2} \). The constructive example builds a situation in which, at equilibrium, a free family of \( S-J+1 \) gradients are present in the economy and there is an exact fraction of \( \frac{1}{S-J+1} \) of the shareholders at each vertex of the simplex of gradients. Shareholders use their gradients to
evaluate infinitesimal changes of production plans. If the status quo does not satisfy the Pareto criterion (see footnote 8), i.e., it maximizes profit with respect to a normalized gradient outside the simplex, then an unanimously preferred challenger can be found. If it satisfies the Pareto criterion, then it is easy to construct production sets for which an improving change can be found for $S - J$ out of the $S - J + 1$ vertices (by pointing toward one of the faces of the simplex). Hence the result.

This example is a ‘worst-case’ scenario, but it is very instructive. It indeed sheds light on a fundamental aspects which jeopardize preference aggregation through voting: the ‘polarization’ of voting weights. Under the Drèze criterion, a polarization of voting weights as it is constructed in the worst-case scenario (the fact that shareholders are evenly distributed over the $S - J + 1$ vertices) is improbable. In the present context, this would mean that shareholders with insurance needs at odds with one another have the same voting weight within a firm. The Drèze criterion breaks this polarization of welfare weights. Indeed, the welfare weights are the post-trade shares and, in the leading example, these post-trade shares are linear functions of the endowment shocks and preference characteristics: they completely smooth out possible devil-like (i.e., like in the worst case scenario) distributions of initial shares. As mentioned in the introduction, the linearity of this dependence is interpreted as a clientele effect in the sense that investors with different insurance needs sort into different firms. This self-selection process endogenously homogenizes the individual characteristics of the social choice problem, thereby making it easier to solve.

As a first conclusion, one could assert that under conservative expectations markets have a stabilizing effect on voting through the clientele effect, a phenomenon only relevant under the Drèze criterion. However, as shown in the next section, the stabilizing effect of markets is very sensitive to the expectations regime. Indeed, under other expectations regimes markets may worsen the difficulty of aggregating individual preferences through majority voting. Clearly Grossman-Hart criterion is immune to changes in the expectations regime.

4 Market disaggregation of individual preferences

4.1 ‘Exit’ expectations

Suppose now that, at a SME($y_j$) $(x, \theta, q)$, shareholders expect that, within firm $j$, a challenger $y_I^j$ will win a proxy fight against the status quo $y_j$. Then all shareholders expect new prices $q'$ to prevail in the future, and thus they compute how they rebalance their optimal portfolio $\theta'$ and corresponding consumption $x'$, so that $(q', x', \theta')$ is a stock market equilibrium associated with the fixed production plans $(y_1, \ldots, y_{j-1}, y_I^j, y_{j+1}, \ldots, y_J)$. The alternative policy $y_I^j$ offers new insurance opportunities. The shareholders whose insurance needs are not as well covered by $y_I^j$ than they were by $y_j$ will exit, at least partially, from the capital of the firm. They will sell (part of) their shares to shareholders whose
insurance needs are better met by \( y'_j \) than by \( y_i \). This is the way \( \theta \) is rebalanced into \( \theta' \).

If these expectations can be confirmed at equilibrium in the sense that under the voting weight distribution \( \theta_j' \) the challenger \( y'_j \) rallies a high enough majority against the status quo, then the latter is not stable any more, at least under the exit expectations regime. To recover stability, it must be the case that the exit expectations regime is never confirmed at equilibrium.

**Definition 4** \((x^*, \theta^*, q^*, y^*)\) is a \( \rho \)-majority exit-stable equilibrium (\( \rho \)-MeSE) if:

- \((x^*, \theta^*, q^*)\) is a stock market equilibrium with fixed production plans \( y^* \),

- \( \forall j \), \( P^*_j(x, \theta, y^*_j) = \emptyset \), for all for all \( y_i \in Z_j \), for all \((x, \theta, q)\) stock market equilibrium with fixed production plans \((y'_1, \ldots, y'_j-1, y_j, y'_j+1, \ldots, y'_J)\).

Intuitively, one expects that shareholders whose insurance needs are worse covered by \( y'_j \) than by \( y_i \), and thus (partially) exit from the capital of the firm, vote for the status quo against the challenger. Hence, under the ‘one share-one vote’ governance, they loose power in the corporate control mechanism. Thus, the exit effect gives endogenously more voting weight to the shareholders who favor the challenger over the status quo. The exit expectation regime should intuitively gives resonance to the victory of the challenger in the voting process. This property is stated in the following proposition which is proved in the appendix.

**Proposition 3** If a \( \rho \)-majority (exit) stable equilibrium is regular\(^1\) then it is \( \rho \)-majority stable (for conservative expectations).

The extent to which the corporate charter needs to be increased to secure that a \( \rho \)-majority stable equilibrium be exit-stable depends on the case under consideration and will be the object of the section below.

### 4.2 Voting under exit

The object of this section is best illustrated under the assumptions of Section 3 where agents have linear-quadratic utility functions. Indeed a simple geometric analysis can thus be carried out.

Let us keep it to the simplest: there are two states of nature \( S = 2 \) and only one firm \( J = 1 \). Agents have linear-quadratic utility functions with \( \gamma^0 = 1 \) for everybody, the same coefficient \( \gamma^1 \) and equal probabilities over the states of nature. Thus they only receive endowment shocks which are restricted to be on a segment: \( \omega_1 \in [A, B] = \)

---

\(^{13}\)A stock market equilibrium with fixed production plans \((x^*, \theta^*, q^*)\) \( \in \text{SME}(y^*) \) is regular if for all \( j \) and \((y^j_n)\)_n, where \( y^j_n \in Z_j \), such that \( y^j_n \to y^j*_n \), and all \( n \), there exists \((x^n, \theta^n, q^n)\)_n, where \((x^n, \theta^n, q^n)\) \( \in \text{SME}(y^*_1, \ldots, y^*_j-1, y^*_j, y^*_j+1, \ldots, y^*_J) \), such that \((x^n, \theta^n, q^n) \to (x^*, \theta^*, q^*) \) uniformly.
The density of the marginal distribution on endowment shocks in period 1 is supposed to exhibit no aggregate risk: \( \Omega_1 = 0 \). Finally, the efficient production set of the firm is \( Z = \{(1, y) \mid y \in \mathbb{R}_+^2 \text{ and } \| y \| = 1 \} \).

Proposition 2 yields, in this simple case, for a proposed production plan \( y \in Z \),

\[
\theta \iota = 1 - y \cdot \omega_1 \quad \text{and} \quad x_\iota = \omega_1 + \theta \iota \cdot y ,
\]

hence \( x_\iota \) is the orthogonal projection of \( \omega_1 \) on the line \( \Delta_y \) tangent to \( Z \) and orthogonal to \( y \). When confronted to the choice between the status quo \( y \) and a challenger \( y' \), under both expectations regime, the same line \( S(y, y') \) separates shareholders preferring \( y \) to \( y' \) (those in \( I(x, \theta, y_1, y'_1) \)) from shareholders preferring \( y' \) to \( y \). \( S(y, y') \) goes through \( C \) and is parallel to the median of the angle between \( y \) and \( y' \): Agents measure consumption bundles through their (euclidian) distance to \( C \). Hence all shareholders \( \iota \) with \( \omega_1 \in [A, \bar{\omega}_1] \) (where \( \bar{\omega}_1 \) is the period 1 endowments of the indifferent shareholders) favor \( y \) over \( y' \). The problem is thus sketched through a Hoteling-like model, as seen from Figure 1.

Under the conservative expectations regime, the distribution of voting weights is fixed to \( f\theta \). When \( y' \) moves on \( Z \), then the separating line \( S(y, y') \) ‘swings’ (going though \( C \)) and there is more or less ‘shares’ left to the status quo \( y \). Hence, as stated in Lemma 1,

\footnote{This remains true for any support of period 1 endowment shocks (even bi-dimensional).}
the best strategy for \( y' \) is to come as close as it can to \( y \) from the right side (on Figure 1 it is from above). This way, it reduces the support of \( y \) to shareholders on the segment \([A, \omega_1^{y \to y}]\). Accordingly, the best initial position \( y^* \) for the status quo is to settle down at the median of the distribution \( f_{\theta} \); this makes it stable with respect to simple majority voting under conservative expectations. If \( f \) is symmetrically distributed around 0, then \( y^* \) is on the forty-five degrees line.

Consider now the 0.5-MSE \( y^* \), and let us check its stability under the exit expectations regime. The important change is that the voting weights, given by the distribution \( f_{\theta} \), now changes with the position of the challenger \( y \) (see Figure 2). The latter is subject to two opposing forces: (i) the ‘swinging effect’ of \( S(y^*, y) \) leaves more or less shares to the status quo \( y^* \), as was seen on Figure 1; (ii) the ‘exit effect’, making agents rebalancing their optimal portfolio, gives more or less relative voting weight to the supporters of the status quo \( y^* \). As easily seen from Figure 2, sliding away from \( y^* \), \( y \) looses support through the swinging effect: from \([A, 0]\) to \([A, \omega_1]\). But on the other hand, it gains support through the exit effect, since the quantity of shares bought by shareholders in \([A, \omega_1]\), measured by the distance to \( \Delta_y \), increases in relative terms. The ‘clientele effect’ under the exit expectations regime gives a better chance to the challenger to defeat the status quo.

Which of the two effects dominates depends on the parameters of the model. Three examples can be underlined.

1. For \( \gamma^1 = 1 \) and \( f \) uniform on \([A, B]\), \( y^* = (\sqrt{2}, \sqrt{2}) \) (on the forty-five degrees line) is stable under the lowest super majority rule for both expectations regime: 0.5 for
conservative expectations; 0.53 for exit expectations: one does not have minimal differentiation anymore. In other words, one needs to raise the corporate charter up to 0.53 to secure that the 0.5-MSE is exit-stable. But then only conservative expectations regime can be self-fulfilling, and therefore, quite paradoxically, no challenger can raise more than 50% of the shares against the status quo: the corporate charter is not even approached!

2. It is easily seen that, for any fixed initial distribution of endowment shocks $f$, there exist a $\gamma^1$ high enough such that the swinging effect always dominates the exit effect. Indeed, the exit effect, based on the computation of $\theta$, does not depend on $\gamma^1$; and the swinging effect does, and is huge when $\gamma^1$ is huge. Hence for $\gamma^1$ high enough, the unique 0.5-MSE is 0.5-majority exit-stable.

3. On the contrary, it can be the case that the unique 0.5-MSE is not $\rho$-majority exit-stable for any rate $\rho < 1$. Indeed, consider the family of symmetric densities $f \equiv e^{2n}$, for an integer $n$, with $e \in [-1, 1]$ and $\omega_1 = (-e, e)$. For all $n$, $y^*$ is the unique 0.5-MSE. For any $\rho \in [0.5, 1]$, there exists $\bar{n}$ such that for $n > \bar{n}$ $y^*$ is not $\rho$-majority exit-stable. Indeed, for a very high $n$, shareholders are concentrated on $A$ and $B$. Then sliding away from $y^*$ (e.g., downward) $y$ keeps the support of shareholders close to $A$, and these shareholders acquire more and more shares as $y$ slides downward. Down to the point where, almost reaching the point of maximal differentiation $(1, 0)$, shareholders in a small neighborhood of $A$ have a quantity of shares close to 2 and shareholders in a small neighborhood of $B$ have a quantity of shares close to 0 (as $\Delta_y$ comes very close to $B$): almost half of the shareholders are still supporting $y$ against $y^*$ (the interval $[A, \bar{\omega}_1]$ still contains a neighborhood of $A$ where there is a high density of shareholders) and have almost all the shares.

Along this last line, an example could be constructed where, between two alternatives $z_1$ and $z_2$, if $z_1$ is the status quo then there is a 99%-majority of the electorate in favor of $z_2$, and if $z_2$ is the status quo then there is a 99%-majority of the electorate in favor of $z_1$. Hence a Condorcet cycle of length two, even for a super majority rule close to unanimity! This worsens by far the extent of the traditional social choice paradoxes. A situation which of course cannot occur under the Grossman-Hart criterion, due to the absence of the exit effect.

5 Applicability of the central result

The object of this section is to extend the applicability of Proposition 1 to a wider class of economies than the leading example. We restrict attention to the Drèze criterion for the sake of simplicity, although the results obtained hold for all the governances described in Section 3.
A condition of Proposition 1 is that the set of normalized equilibrium gradients is convex. Since we aim here at linking that condition to assumptions on the distribution of primitive characteristics, we are naturally led to consider convex sets of primitive characteristics and gradients which are linear with respect to primitive characteristics and choice variables. An obvious class is the set of full-fledged quadratic utility functions.

For parameters \((\gamma, \zeta) \in \mathbb{R}^{S+1} \times \mathbb{R}^{(S+1)^2}\), consider the utility function of the form:

\[
    u(x) = \sum_{s=0}^{S} \gamma^s x^s + \sum_{s=0}^{S} \sum_{s'=0}^{S} \zeta^{ss'} x^s x^{s'} = \gamma \cdot x + x^t Z x.
\]

This extends the quadratic feature of the leading example to period zero and does not imply separability with respect to time and states. We make the implicit assumptions on the parameters \((\gamma, \zeta)\) such that \(u\) fulfills the assumptions of Section 2 on the relevant domain. The parameters \(\zeta\) are the same across agents. This family of utility functions would allow us to provide a ‘local’ version of the central result by considering perturbations of an economy with identical agents where, broadly speaking, all consumers’ utility functions have the same curvature and receive a \(\sigma\)-concave shock on their endowments, marginal rates of substitution and initial portfolios.

The set of primitive characteristics \(I\) contains vectors of parameters: \(\iota = (\omega, \delta, \gamma)\).

**Proposition 4** Suppose that \(I\) is compact and convex, and moreover endowed with a \(\sigma\)-concave density, \(f\). Then at a SME(\(\eta\)), \(E\), the induced probability distribution \(\eta\) on \(N\) has compact and convex support and is \(\sigma/[1 + (2S + J + 3)\sigma]\)-concave; consequently \(E\) is a 0.64-majority stable equilibrium.

**Proof:** First observe that gradients are linear in consumptions and characteristics: for all \(x\),

\[
    Du_\iota(x) = \gamma + (Z + Z^t)x.
\]

Therefore, at equilibrium, consumptions and shares are linear functions of the primitive characteristics. Indeed, consider two agents \((\iota, \iota')\); at equilibrium, \(Du_\iota(x)\) and \(Du_{\iota'}(x')\) are orthogonal to the transfer matrix, \(\left(\begin{array}{c} -q \\ Y \end{array}\right)\). Since, for all \(t \in [0, 1]\),

\[
    Du_{\iota + (1-t)\iota'}(tx + (1-t)x') = tDu_\iota(x) + (1-t)Du_{\iota'}(x')
\]

the gradient of agent \(t\iota + (1-t)\iota'\) at \(tx + (1-t)x'\) is orthogonal to the transfer matrix; moreover, it is easily checked that by taking \(\theta_{\iota + (1-t)\iota'} = t\theta_\iota + (1-t)\theta_{\iota'}\), the budget constraints of agent \(t\iota + (1-t)\iota'\) are satisfied. Hence agent \(t\iota + (1-t)\iota'\) satisfies the first order conditions of her/his optimization program by consuming \(tx + (1-t)x'\).

The relevant distribution to consider on the space of parameters \(I\) is \(f\theta_j^+\), for firm \(j\). Given the linearity of \(\theta_j\), the same line of argument as in the proof of Corollary 1 shows that \(f\theta_j^+\) is \(\sigma/(\sigma + 1)\)-concave on a compact convex support.
We turn now to the density $\eta_j$, induced by $f\theta_j^+$ on $\mathcal{N}$. Fix two normalized gradients $\nabla$ and $\nabla' \in \mathcal{N}$. Their pre-images in $\mathcal{I}$, denoted $L$ and $L'$, are a linear subspace: indeed, e.g., $L$ is the pre-image through the (linear) gradient mapping $Du$ of a line in $\mathbb{R}^{S+1}$. For all $t \in [0,1]$, given the convexity of $\mathcal{I}$ the pre-image, $Lt$, of $t\nabla + (1-t)\nabla'$ in $\mathcal{I}$ contains the Minkowski average of $L$ and $L'$: $tL + (1-t)L'$. Hence
\[
\int_{Lt} f(\iota)\theta_j^+(\iota) d\iota \geq \int_{tL + (1-t)L'} f(\iota)\theta_j^+(\iota) d\iota
\]
The Prékopa-Borell theorem (see, e.g., Caplin & Nalebuff (1991)) gives:
\[
\int_{tL + (1-t)L'} f(\iota)\theta_j^+(\iota) d\iota \geq \left[ t \left( \int_{L} f(\iota)\theta_j^+(\iota) d\iota \right)^\phi + (1-t) \left( \int_{L} f(\iota)\theta_j^+(\iota) d\iota \right)^\phi \right]^{1/\phi}
\]
where $\phi = \sigma/[1 + (n + 1)\sigma]$ and $n = 2S + J + 2$ stands for the dimension of the set of parameters $\mathcal{I}$. Hence the result. $\square$

If the maximization program of a consumer is solved, then it is seen that the clientele effect is present in the sense that agents with similar gradients before trade, choose the same portfolio even though they may have different initial shares. Therefore, markets tend to smooth out any devil-like initial distribution of shares just as in the leading example.

References


Appendix

Product of a $\phi$-concave and a $\psi$-concave distribution

**Lemma 2** If $G : K \to \mathbb{R}_+$ is $\phi$-concave and $H : K \to \mathbb{R}_+$ is $\psi$-concave. Then $F : K \to \mathbb{R}_+$ defined by $F(a) = G(a)H(a)$ for all $a \in K$ is $\nu$-concave for all

$$\nu \leq \frac{\phi \psi}{\phi + \psi}.$$ 

**Proof:** It follows from the definition of $\nu$-concavity that if

$$(((1 - t)G(a)\phi + tG(b)\phi)^{1/\phi}((1 - t)H(a)\psi + tH(b)\psi)^{1/\psi})^\nu$$

$$\geq (1 - t)(G(a)H(a))^{\nu} + t(G(b)H(b))^{\nu}$$

for all $a, b \in K$ and $t \in [0, 1]$ then $F : K \to \mathbb{R}_+$ is $\nu$-concave.

Let $g, h : [0, 1] \to \mathbb{R}_+$ be defined by

$$g(t) = ((1 - t)G(a)\phi + tG(b)\phi)^{1/\phi}$$

$$h(t) = ((1 - t)H(a)\psi + tH(b)\psi)^{1/\psi}$$

then $g$ is $\phi$-concave and $h$ is $\psi$-concave. Let $f : [0, 1] \to \mathbb{R}_+$ be defined by $f(t) = (g(t)h(t))^{\nu}$ then the second-order derivative is

$$D^2 f = \nu(gh)^{\nu - 2}((\nu - 1)((gDf)^2 + (fDg)^2) + 2\nu(gDf)(fDg)$$

$$+fg(gD^2f + fD^2g))$$

$$\leq \nu(gh)^{\nu - 2}((\nu - \phi)(gDf)^2 + 2\nu(gDf)(fDg) + (\nu - \psi)(fDg)^2).$$

The “$\leq$” follows from the fact that $g$ being $\phi$-concave is equivalent to $g^{\phi}$ being concave so $D^2 g^{\phi} = \phi g^{\phi - 2}((\phi - 1)(Dg)^2 + gD^2 g) \leq 0$ implying $gD^2 g \leq (1 - \phi)(Dg)^2$ – similarly for $h$ and $\psi$.

Finally $(\nu - \phi)(gDf)^2 + 2\nu(gDf)(fDg) + (\nu - \psi)(fDg)^2 \leq 0$ for all values of $gDf$ and $fDg$ if and only if $\nu \leq \phi \psi / (\phi + \psi)$. Hence, $F : K \to \mathbb{R}_+$ is $\nu$-concave for all $\nu \leq \phi \psi / (\phi + \psi)$. $\square$

**Proof of Proposition 3**

Let $(x^*, \theta^*, q^*, y^*)$ be a regular $\rho$-majority (exit) stable equilibrium. Then for all $(y^n_i)_n$, where $y^n_i \to y^n_i^*$, there exists $(x^n, \theta^n, q^n)_n$, where $(x^n, \theta^n, q^n)$ is an equilibrium with fixed production plans $(y^n_1, \ldots, y^n_i, y^n_{i+1}, \ldots, y^n_J)$, such that $(x^n, \theta^n, q^n) \to (x^*, \theta^*, q^*)$. Therefore, if $1/||y^n_i - y^n_i^*||/||y^n_i - y^n_j^*|| \to t_i$ and $\theta_{i,j} \nabla_i \cdot t_j > 0$, then there exists $N$ such that if $n \geq N$, then $u_i(x^n + \theta^n_{i,j}(y^n_j - y^n_i^*)) > u_i(x^n_i)$, because $(x^n, \theta^n, q^n) \to (x^*, \theta^*, q^*)$ and because the utility function is continuous. Hence if a $\rho$-majority (exit) stable equilibrium is regular, then it is $\rho$-majority stable (for conservative expectations). $\square$