On the Suboptimality of First-Price Rules*

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Abstract

Buyers engaged in procurement can face substantially differentiated sellers. The need to balance pricing concerns and preferences over sourcing leads to the use of handicapping rules, whereby preferred sellers enjoy a formal advantage in the procurement rules. Given the prevalence of such mechanisms it is of substantial practical and theoretical importance to understand and rank the outcomes of different mechanisms. This paper analyzes first price auctions with concave handicapping rules, which are the most common means of skewing auction by buyers who have a preferred subset of sellers. We find conditions under which any first price auction with concave handicapping is ex-post dominated by a simple second-price auction design or its open equivalent.

Keywords: Asymmetric Auctions, Handicapping Rules, Differentiation, Mechanism Design, First Price Auctions, Second Price Auctions, Procurement, ρ-concavity.

1 Introduction

In procurement settings it is a common occurrence for the buyer to have a preferred supplier.1 From a business perspective this can be the result of quality or reliability differences among the set of competing suppliers. Incumbency status or technological switching costs are often referred as common reasons for skewing the rules in favor of one supplier or another.

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1All the results have an obvious albeit less intuitive counterpart in auctions where a seller has preferences over buyers.
Many international and government institutions adopt a form of discrimination based on the identity of the bidder in order to promote larger policy goals. The World Bank (2004a, 2004b), for example, requires loan recipients in developing countries to favor local sourcing in the procurement rules and adopts a scoring system to favor some providers of consultant services over others. These rules have, in the meantime, been widely adopted as "good practice" in many of the sourcing processes of both non-profit agencies as well as corporations. The federal government through its various agencies has adopted preferential procurement rules to favor minority and women owned businesses (for general procurement see Cabral and Greenstein (1990), Corns and Schotter (1999), for federal highway contracts see Marion (2006 and 2007), and for PCS auctions see Cramton (1995)). All of these attempts at skewing the allocation represent the balancing in the procurement process of price competition against the probability of transacting with the preferred seller.

Auction theory has long recognized that in the presence of asymmetries, the optimal mechanism need not be symmetric (see Myerson (1981), Maskin and Riley (1981), McAfee and McMillan (1988)). Unfortunately, optimal mechanisms are highly sensitive to informational assumptions and can face serious legal and practical roadblocks in implementation even by an informed designer. Alternative mechanisms, usually in the form of scoring rules have been proposed as a solution to the problem (see Che (1993), Arozamena and Cantillon (2004)). The optimal scoring rule distorts quality downward, and can be implemented by either the first or second score rules. Optimal scoring rules are also burdened by informational assumptions and have been roundly criticized on these grounds (see Wolfstetter and Lengweiler (2006)). In practice, it is the mechanisms with simple handicapping or scoring rules that dominate the procurement landscape where transparency plays a far greater role.

\footnote{Branco (1997) adds common value aspects and correlation to costs. Naegelen (2002) extends this result by allowing for an exogenous preference for one bidder. Asker and Cantillon (2006) expand the results to multi-dimensional quality and discuss the connection to scoring rules. Manelli and Vincent (1995) also study procurement settings. They consider situations in which low costs are correlated with low quality. This paper differs from theirs in that the buyer’s relative valuation for purchasing from the two sellers is fixed and known.}

\footnote{In Gannza and Peclivanos (2000) the buyer can affect the degree to which one firm or another has an advantage, by designing the object of the procurement process. When the mechanism can be asymmetric, the buyer will choose a design which favors one firm and recapture that advantage via the mechanism. If the buyer is constrained to symmetric mechanisms, he will choose a design that increases homogeneity, and thus competition. See also Rezende (2004) for interesting findings on the role of the seller’s uncertainty about his own preferences.}
larger role than optimization. The main argument of this paper is that even within the class of “simple” mechanisms, we can make relatively strong arguments for one design over others.

In practice, the most prevalent method of skewing the allocation is a first price design (pay your bid) with a handicapping rule (FP-HR) whereby the preferred supplier(s) can win the contract by coming only within a fixed or proportional margin of the lowest bid. The World Bank requires for the local sourcing contracts it finances a percentage handicap of up to 15% in favor of local suppliers. The federal government uses a similar scheme with a percentage handicap of 5% in favor of minority businesses in highway procurement contracting. In the PCS auctions, minority and women owned businesses were given a 25% bidding credit on certain licenses, “designed to offset any discrimination these firms may face in raising capital and offering PCS services” (Cramton 1995). In corporate procurement, quasi-linear scoring rules are the most common form of “rewarding” incumbent suppliers. When quality is exogenous, such mechanism come mainly in the form of first price auctions with fixed handicaps (FPFA). Alternatively, one can imagine second-price or open designs. In one of these Second Price Bonus Auctions (SPBA) the low bidder wins the contract and preferred suppliers receive, on top of the payment based on the losing bid, a bonus upon winning. World Bank rules on allocating consulting contracts function in this spirit where the bonus is determined by (a rather opaque) quality scoring process.

Cornes and Schotter (1999) conducted experiments to investigate the effects of affirmative action programs in procurement. They show that small percentage handicaps improve the price performance of auctions in asymmetric environments. Shachat and Swarthout (2003) provide further experimental evidence that the optimal English auction with a bonus outperforms a standard sealed bid request for proposal in a setting with uniformly distributed costs. Cabral and Greenstein (1990) discuss the empirical implications of favored bidding in federal procurement, while Wolfstetter and Lengweiler (2006) discuss favoritism and corruption, in the context of bonus auctions. Marion (2006) empirically estimates the price effect of favoring disadvantaged bidders through proportional bonuses and (2007) points out that bid preferences can have significant negative participation effects on disadvantaged bidders. While handicapping rules like these can have non-negligible participation effects it not the main focus of this paper. Neither is the com-

\textsuperscript{4}Flambard and Perrigne (2006) provide empirical evidence form Canadian snow removal procurement auctions supporting this conclusion. See also Schotter and Weigelt (1992) for experimental evidence in handicapped tournaments.
parison of mechanisms with handicapping rules to standard auctions, since one generally lacks a full description of the preference of the buyer. The main effort goes into exploring the efficacy of alternative designs. This is seldomly fully explored in the literature because of the intricacies of deriving the a full equilibrium characterization of the notoriously intractable asymmetric first price auctions.

A good entry point into the general literature on asymmetric first price auctions is Maskin and Riley (2000a,2000b). Lebrun (1996, 1999) has done extensive work on the existence and characterization of equilibria in these settings. The equilibrium in these auctions is unique under log-concave distributions and various other specifications (Lizzeri and Persico (2000), Rodriguez (2000), Maskin and Riley (2003), Lebrun (2006)). A key result on existence of monotone strategies in first price auctions is provided by Reny and Zamir (2004).5

This paper argues that an equally simple approach to the preferred bidder problem can be found in second price bonus auctions (SPBA) or their open equivalents. A preferred bidder in an SPBA will win the contract if his is the lowest bid. He receives on top of the payment based on the losing bid(s) a pre-specified bid-independent bonus whenever he wins. His opponent(s) do not receive such a bonus upon winning the auction. This design will still skew the allocation rule towards the preferred bidders but has a two-fold advantage over first-price handicapping rules. Mares and Swinkels (2010a) show that SPBAs outperform simple handicapping rules which are implemented using first price mechanisms.6 Further, Mares and Swinkels (2010b) show that SPBAs manage to implement something which is indeed very close to the optimum based on limited informational assumptions. This paper takes one further step and shows that SPBA implementations outperform *ex-post* anything that can be implemented via a set of complicated handicapping rules using a first price design. This includes proportional handicaps like those used by the World Bank and the federal government or corporate procurers, or combinations of proportional and fixed handicapping rules.7

The paper proceeds as follows. Section 2 describes the basic modeling

5 For a further summary of the literature on asymmetric first price auctions, see Krishna (2002) and the references therein.

6 In particular, they concentrate on first price handicap auctions (FPHA), where a preferred bidder needs only to come within a fixed, pre-specified amount of other bids in order to win.

7 See Kirkegaard (2010) for an interesting analysis of such combinations in a tournament setting.
approach and derives some basic results on the shape of optimal allocations. Section 3 formally introduces the allocation generated by FP-HRs and derives some basic characterizations. Section 4 states the main ranking result and its intuition. Section 5 studies the geometric properties of FP-HR allocations and provides the key result on bounds for slopes. Section 6 discusses possible extensions of the central result. Section 7 concludes. All lengthy or non-essential proofs are relegated to the appendix.

2 The Model

Consider a situation where a buyer faces two sellers who independently draw costs $c_1$ and $c_2$ from some distribution $F$ with strictly log-concave density $f$. Denote by $\bar{F} = 1 - F$ the reverse cumulative and assume $f$ has support $[0, 1]$ and is twice continuously differentiable. At any given price $p$ the buyer’s utility from purchasing from supplier $i$ is

$$U_i(p) = v_i - p.$$ 

Assume that $v_1 - v_2 = \Delta > 0$, that is, supplier 1’s product is preferred by the buyer by the amount $\Delta$, which is known to the buyer at the time he commits to the auction rules.

For any positive twice differentiable function $g$ with support $[0, 1]$ define

$$W_g(c) = \frac{g''g}{(g')^2}(c),$$

a measure the local $\rho$-concavity of $g$.

2.1 First Price Auctions with Handicap Rules

A First Price Auction with Handicapping Rule $\theta$ (FP-HR$_\theta$) requests bids $b_1$ and $b_2$ from the two competing suppliers. The seller commits himself

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8Strict log-concavity is required for the sole purpose of ensuring that $f$ is strictly monotone. The interested reader can find an extensive discussion on the relaxation of the symmetry assumption in Mares and Swinkels (2010a).

9$\Delta$ can be assumed to be common knowledge although this is immaterial for the discussion that follows.

10For the interpretation of $W_g$, its link to the Prekopa-Borell theorem and applications to auction theory and mechanism design see Mares and Swinkels (2010a). Anderson and Renault (2003) and Fabinger and Weyl (2009) use a similar notion to measure the concavity of demand curves in their work on oligopoly pricing. They note that $W_g$ appears in the economic literature as early as Cournot’s 1838 work on oligopolies. See also Caplin and Nalebuff (1991a,1991b) for a related discussion on local $\rho$-concavity in a social choice context.
to an allocation rule determined some strictly increasing, concave function \( \theta \in C^2[0,1] \), with \( \theta(b) > b \) for \( b > 0 \), whereby the favored bidder receives the contract iff

\[
\begin{align*}
 b_1 < \theta(b_2).
\end{align*}
\]

Whenever \( b_1 > \theta(b_2) \) the contract is awarded to bidder 2. Ties are inessential since they occur with probability zero in equilibrium.\(^{11}\) The winning bidder is compensated in the amount of his bid but no more than 1 - the highest possible cost.\(^{12}\) First Price Handicap Auctions analyzed extensively in Mares and Swinkels (2010a) simply define \( \theta(b) = b + A \), for some \( A > 0 \); while World Bank and government procurement rules define \( \theta(b) = Ab \), for some \( A > 1 \).

### 2.2 Second Price Bonus Auctions

An alternative simple mechanism which skews the allocation in favor of the preferred bidder has sellers submitting bids \( b_1 \) and \( b_2 \) and allocates to the lowest bidder. If the winning bidder is the preferred supplier his payment will \( \max(b_2 + A, 1) \) whereas supplier 2 receives upon winning the contract a payment of \( \max(b_1, 1) \). The bonus \( A \) is part of the auction design and thus subject to the seller’s commitment. It is easy to see that a dominant strategy equilibrium exists in this game where \( \beta_1(c) = c - A \) and \( \beta_2(c) = c \), which results in an allocation whereby the preferred supplier wins whenever

\[
\begin{align*}
 c_1 < c_2 + A.
\end{align*}
\]

There are several advantages of this simple mechanism. First, there are multiple ways of implementing the same allocation via open auctions. Second, in contrast to any FP-HR, bidding in SPBA’s is straight-forward. Thirdly, Mares and Swinkels (2010b) provide methods to construct a well-performing bonus \( A(\Delta) \) which is relatively detail-free.\(^{13}\) Finally, Mares and Swinkels (2010a) show that any FPHA will lead to an allocation which can be improved upon by an SPBA with a well-chosen bonus. The ultimate goal of this paper is to extend this result and find conditions under which any FP-HR is \textit{ex-post} dominated by an appropriately chosen SPBA.

\(^{11}\)See Jackson and Swinkels (2005) for details.

\(^{12}\)This is a non-traditional role of reserve prices. The aim is to normalize the utility of the worst cost type to zero. As a consequence, the buyer refrains from any exclusion. Effectively, no such exclusion will be optimal if \( c_2 \) is large and \( f(1) > 0 \). For more details on the role of reserve prices and simple handicapping rules see Mares and Swinkels (2010b).

\(^{13}\)Mares and Swinkels (2011) provide a full characterization of the optimal SPBA when distributional information is available.
2.3 Optimal Mechanisms

Let
\[ \omega(c_i) = c_i + \frac{F(c_i)}{f(c_i)} \]
be the virtual cost of \(i\). Because \(F\) is log-concave, \(\omega' \geq 1\).\(^{14}\) Note also that \(\omega'(c) = 2 - W_F(c)\) so that convex (concave) virtual costs are characterized by decreasing (increasing) \(W_F\).

Consider any deterministic mechanism in which the buyer always buys. From incentive compatibility, any such mechanism is characterized by an increasing function \(\eta\) such that 1 wins if and only if \(c_1 < \eta(c_2)\). Adapting Myerson (1981) or Riley and Samuelson (1981) in the obvious manner, the buyer’s expected surplus from mechanism \(\eta\) is

\[ BS(\eta) = v_2 - 1 + \int \int I_{\{c_1 < \eta(c_2)\}} (\Delta - \omega(c_1) + \omega(c_2)) f(c_1) f(c_2) dc_1 dc_2. \] (1)

This is intuitive. Always buying from 2 gives the buyer surplus \(v_2 - 1\), since 2 must receive 1 if he is to sell for all \(c_2\). The integral represents the change in buyer surplus from buying from 1 according to \(\eta\). So, among mechanisms that always buy, 1 optimally wins if \(\Delta > \omega(c_1) - \omega(c_2)\) and 2 wins otherwise. Let \(\eta_M\) given by

\[ \Delta = \omega(\eta_M(c_2)) - \omega(c_2) \] (2)

be this mechanism. Note also that as long as \(f(1) > 0\), \(\eta_M(1) < 1\).\(^{15}\)

Characterizing the slope of \(\eta_M\) is also a simple exercise.

Lemma 1 If \(W_F\) is decreasing, then \(\eta'_M \leq 1\). If \(W_F\) is increasing, then \(\eta'_M \geq 1\).

To see this, differentiate (2) and note that since \(\omega\) is increasing \(\eta_M(c_2) > c_2\), and that

\[ \eta'_M(c_2) = \frac{2 - W_F(c_2)}{2 - W_F(\eta_M(c_2))} \]

\(^{14}\)This follows directly form the assumption that \(f\) is log-concave (see for example Karlin (1968)).

\(^{15}\)This condition implies that virtual costs are bounded and thus for large enough \(v_2\) no exclusion is optimal. Conversely, when \(f(1) = 0\) virtual costs diverge for inefficient types and thus some exclusion will always be optimal. Regardless of these considerations, the question of whether some exclusion is optimal is immaterial to the choice between first and second-price designs.
yielding the result.\textsuperscript{16} So, if \( W_F \) is a constant, then \( \eta'_M = 1 \). In this case, the optimal mechanism can be directly implemented by an appropriate \( SPBA \). This is true for the power distributions, \( F(c) = e^c \), when the bonus is set at \( A = \frac{\alpha \Delta}{\alpha + 1} \).

\section{Allocations in First Price Auctions}

Consider an equilibrium of the FP-HR\( _\theta \) with \( \beta_1 \) and \( \beta_2 \) increasing strategies. Such an equilibrium exists by Reny and Zamir (2004) and is strictly monotone.\textsuperscript{17} Define the resulting allocation function \( \phi \) by the implicit equation

\[ \beta_1(\phi(c_2)) = \theta(\beta_2(c_2)). \]  

(3)

Observe that the monotonicity of \( \beta_1 \) is sufficient to make this a well-defined equation for any \( c_2 \) where \( \phi(c_2) \in [0, 1] \). Under these conditions \( \phi \) itself is monotone increasing as a composition of increasing functions and thus admits an inverse \( \psi \).\textsuperscript{18}

Observe that since \( \beta_1(1) = 1 \) a non-preferred supplier of type \( \bar{A} = \theta^{-1}(1) \) cannot guarantee himself positive surplus.\textsuperscript{19} Thus \( \phi : [0, \bar{A}] \rightarrow [0, 1] \) is well-defined and is easily interpretable in the type-space. Bidder 1 wins the asset if \( c_1 \leq \phi(c_2) \) while bidder 2 is the winner whenever \( c_1 \geq \phi(c_2) \).\textsuperscript{20} The rest of this section shows how \( \phi \) ties down the equilibrium behavior of bidders. The other results in this paper will exclusively concentrate on the properties of \( \phi \).

Note first that under any increasing equilibrium of FP-HR\( _\theta \) \( \phi(0) = 0 \). To see this consider a situation where bidder 1 with type 0 contemplates a bid \( B_1 < \theta(\beta_2(0)) \). Increasing the value of his bid fractionally entails no loss in the probability and strictly increases his payoff. Conversely, consider a situation where bidder 2 with type 0 contemplates a bid \( B_2 \), such that

\textsuperscript{16}An example of increasing log-concave density which yields strictly convex virtual costs is \( f(x) \propto 1 + 2x \), while strictly concave virtual costs result from \( f(x) \propto x \ln(1 + x) \).

\textsuperscript{17}The formulation of Reny and Zamir (2004) explicitly allows for FP-HR if \( \theta \) is strictly monotone but offers only weak monotonicity of equilibria. In a similar formulation Rodriguez (2000) proves that in a two-bidder environment any piece-wise monotone equilibrium is strictly increasing over the range of winning bids. A similar conclusion would result from Lizzeri and Persico (2000) but their approach excludes private value models.

\textsuperscript{18}If \( \beta_1(e_1) > \theta(\beta_2(e_2)) \) for all \( e_1 \) define \( \phi(e_2) = 0 \) and analogously define \( \phi(e_2) = 1 \) if \( \beta_1(e_1) < \theta(\beta_2(e_2)) \) for all \( e_1 \).

\textsuperscript{19}We rule out pathologies where the bidding of types higher than \( \bar{A} \) affect the equilibrium behavior of the preferred bidder.

\textsuperscript{20}As usual, ties are inessential since they occur with zero probability.
\( \theta(B_2) < \beta_1(0) \). Then \( B_2 \) wins with probability 1 allowing for a marginal increase which increases expected profits strictly. This simple observation shows that in equilibrium, bidding undoes handicapping at least for the most efficient types.

Next note that the first-order characterization of equilibrium bidding in this environment is fully described by the allocation \( \phi \).

**Theorem 1** Let \( \beta_1, \beta_2 \) be an increasing equilibrium of FP-HR\( \theta \). Then \( \phi(0) = 0 \), and \( \phi(\bar{A}) = 1 \). Player 1’s surplus with cost \( c \) is

\[
S_1(c) = \int_c^1 F(\psi(s))ds. \tag{4}
\]

Player 2’s surplus with cost \( c \) is

\[
S_2(c) = \int_c^A F(\phi(s))ds. \tag{5}
\]

If \( f \) and \( \theta \) are \( C^k \), then \( \beta_1, \beta_2, \) and \( \phi \) are \( C^{k+1}[0, \bar{A}] \). On their domains

\[
\beta'_1(c) = W_{S_1}(c) > 0, \tag{6}
\]

\[
\beta'_2(c) = W_{S_2}(c) > 0, \tag{7}
\]

and

\[
\phi'(c) = \frac{1}{\theta'(\beta_2(c))} S_1(\phi(c)) S_2(c) \frac{f(\phi(c))}{F_2(\phi(c))} > 0. \tag{8}
\]

While of great use in characterizing the allocation \( \phi \) the result above has one major drawback. The local behavior of the slope of the allocation function \( \phi'(c) \) depends explicitly on the entire equilibrium behavior of types higher than \( c \).\(^{21}\) The novelty of the approach in this paper is to bound \( S_1 \) and \( S_2 \) to obtain qualitative features of \( \phi' \).

## 4 Ranking Auction Designs

In determining the relative merits of FP-HRs and SPBAs we will concentrate on an ex-post criterion. The structure of the problem lends itself naturally

\(^{21}\)This is common to all first-order approaches to first price auctions. See Lebrun (1996, 1998), Corns and Schotter (1999), Maskin and Riley (2000a) and Lizzeri and Persico (2000).
to a geometric interpretation. In the type-space, the optimal allocation $\eta_M$ lies everywhere above the main diagonal ($\eta_M(c_2) > c_2$). This prescribes, as expected, a distortion towards the favored bidder. Its slope is dependent on the local curvature of the distribution measured by $W_F$. A decreasing $W_F$ (concave virtual costs) results in $\eta'_M < 1$. Intuitively, optimality in this case requires that if the unfavored bidder wins for certain cost realizations he should still win if both costs increase by the same amount, or, in other words, progressively less distortion is needed for less efficient types.

In contrast, for any FP-HR$_{\theta}$ we have already established that the induced allocation is characterized by $\phi(0) = 0$. This essentially means that any distortion the seller might have wanted to induce is undone by the equilibrium bidding, at least for the most efficient types. Further, $\phi(\bar{A}) = 1$ shows that at least for inefficient types any FP-HR$_{\theta}$ over-distorts whenever $\eta_M(1) > \bar{A}$, since no unfavored bidder with cost higher than $\bar{A}$ ever stands a chance of winning. So, at least, for a large class of designs FP-HRs seem to work in the opposite direction of the Myersonian prescription.

Second Price Bonus Auctions are easier to understand since they implement lines which are parallel to the main diagonal, and thus induce an even distortion. The identity of the winner remains the same if both bidders cost increases by the same amount. In particular we have already seen that for a class of distributions SPBAs implement the Myersonian optimum exactly. This idea highlights the geometric intuition of the main ranking if the slopes of the allocations can be proven to satisfy the inequality

$$\eta'_M < 1 \leq \phi',$$

then an appropriately chosen SPBA implementing $\lambda$ improves upon any FP-HR$_{\theta}$ point-by-point as in Figure 1.

The inequality guarantees at most a single crossing between $\phi$ and $\eta_M$. If no such crossing exists it easy to see that $\lambda$ can be chosen to lie strictly between $\phi$ and $\eta_M$, while and still implementing an ex-post improved allocation. Lemma 1 gives sufficient conditions for the first part of (9) and thus we have the main result of the paper.

**Theorem 2** If $W_F$ is decreasing then for any FP-HR$_{\theta}$ with $\phi' \geq 1$ there exists an SPBA which performs better ex-post.

The usefulness of this particular result rests on the fact that no explicit characterization of $\phi$, and thus of the equilibrium, is required. A global bound on $\phi'$ is sufficient. In light of this discussion, the rest of the paper will focus on deriving a set of sufficient conditions for obtaining such bounds.
In contrast to the existing literature which adopts ex-ante criteria for the ranking of auction types our result offers a clear-cut rationale for SPBAs or their open counterparts. The inherent distributional assumptions are a direct consequence of this approach which combines ex-post ranking with generally specified handicapping rules.

This result does not imply that any SPBA dominates any particular FP-HR\textsubscript{θ}. On the contrary, particular FP-HRs can be shown to yield higher surplus than specific SPBAs.\textsuperscript{22} It rather shows that the optimal SPBA outperforms ex-ante any FP-HR\textsubscript{θ}, which means that buyers can only profit by focusing their design efforts towards this class of mechanisms.

5 The Slope of Allocation

The starting point for any characterization of $\phi'$ is Theorem 1. The problem with the characterization based on first order conditions is that (8) has the slope of the allocation function depending explicitly on the behavior of the equilibrium to the right of it. Any attempt to obtain a universal bound on the slope $\phi'$ must overcome this problem. On the other hand, Theorem 1 provides the differentiability of $\phi'$ itself and thus allows us to explore the properties of interior minima and maxima of $\phi'$ by looking at the behavior of $\phi''$ at such points. This is precisely what the next result exploits. The next subsection analyzes in greater detail the behavior of $\phi'$ at the boundary of the domain in order to guarantee the existence of interior minima. With these two results in hand we can then proceed to rule out by contradiction the possibility that $\phi'$ ever falls below 1.

We begin our analysis by the determination of a bound for the case when $\phi'$ admits an interior global minimum $r \in (0, \bar{A})$ so that $\phi'(r) \leq 1$.

**Theorem 3** Let $r$ be an interior global minimum of $\phi'$. Assume that $\theta$ is concave, $\phi'(r) \leq 1$, that $\lim \inf_{c \to \bar{A}} \phi'(c) > 1$, and that $\phi(r) \geq r$. Let

$$H(r) = \left( \frac{1}{W_{\hat{f}F}(\phi(r))} - 2 \right) \left( \frac{f}{\hat{F}}(\phi(r)) - \frac{\hat{f}}{\hat{F}}(r) \right) + \left( \frac{f'}{\hat{f}}(r) - \frac{f'}{\hat{f}}(\phi(r)) \right).$$

Then, $\phi''(r) >_{s} H(r)$, where $>_s$ denotes that the LHS is strictly positive whenever the RHS is weakly positive.

\textsuperscript{22}Mares and Swinkels (2010a) show that any FP-HR\textsubscript{θ} with $\theta(b) = b + \Delta$ is isomorphic to a standard auction with shifted asymmetric costs. As such, if one chooses $\Delta = \Delta$ and compares it to the symmetric second price auction, one concludes as in Maskin and Riley (2000a) that the former outperforms the latter in an ex-ante sense.
The power of Theorem 3 is that \( H \) does not depend on the behavior of \( \phi \) after \( r \) but lets us replace a complicated endogenously defined object by an expression tied to the curvature of \( f \hat{F} \) and thus the primitives of the model. The next observation shows the usefulness of this connection.

**Lemma 2** If \( f \) is increasing (decreasing) then \( W_{f \hat{F}}(c) \leq (\geq) \frac{1}{2} \).

This means that whenever \( f \) is increasing the first parenthesis in (10) is positive. If \( \phi(r) \geq r \) then by log-concavity the second and third terms are also positive. Thus \( \phi' \) cannot admit interior global minima where \( \phi'(r) < 1 \), since otherwise the expression in Theorem 3 is positive thereby violating the requirement \( \phi''(r) = 0 \). Thus the search for conditions under which \( \phi' \geq 1 \) boils down to ruling out violations of the condition on the relevant borders and showing that an FP-HR indeed distorts the allocation towards the preferred supplier - i.e. \( \phi(r) \geq r \).

### 5.1 Other Geometric Properties of \( \phi \)

The balance of the section is devoted to understanding the conditions under which Theorem 3 applies. We then use the result to show that 1 is a lower bound on \( \phi' \) thus completing the proof of our main ranking result. To begin our analysis, we begin with some basic properties of \( \phi \).

**Lemma 3** For any FP-HR we have,

- (A) \( \phi(c) \geq c \), for all \( c \in (0, \bar{A}) \),

- (B) if \( \theta'(c) \geq 1 \) then \( \phi(c) < \theta(c) \), for all \( c \in (0, \bar{A}) \).

Simply put the favored bidder takes part of his advantage in the form of a higher margin and part in the form of a higher probability of winning. Bidders are thus undoing in equilibrium some of the distortion designed by the auctioneer.

**Lemma 4** For any FP-HR we have,

- (A) \( \phi'(0) \geq 1 \) and the inequality is strict if \( \theta(0) > 0 \) or \( \theta \) strictly concave,

- (B) \( \liminf_{c \to \bar{A}} \phi'(c) \) > 1.

We thus conclude that \( \phi' \geq 1 \) at each boundary, which means that if \( \phi'(c) < 1 \) for some \( c \) then \( \phi' \) admits a global interior minimum \( r \) with the property that \( \phi'(r) < 1 \).\(^{23}\) These results do not place additional restrictions on the shape of \( f \) beyond log-concavity.

\(^{23}\)Note that the proof of part (A) shows that for the pivotal case \( \theta(b) = Ab \) we have \( \phi'(0) = 1 \).
With this result in hand we can now establish by Theorem 3 and Lemma 2 that \( \phi'(r) > 1 \) for any interior minimum of \( \phi' \), whenever \( f \) is increasing, and thus conclude with the following result.

**Theorem 4** If \( f \) is increasing and \( W_F \) is decreasing then any FP-HR yields an ex-post inferior allocation to that of an appropriately chosen SPBA.

It is interesting to note that the class of increasing densities lies also at the heart of Maskin and Riley’s (2000a) classic result of the superiority of standard first price auctions over their second price counterparts. Secondly, even if the condition on the concavity of virtual costs is violated, Mares and Swinkels (2010b) show, using an independent methodology, that SPBA’s come extremely close to the optimal allocation. In particular for convex virtual costs an SPBA constructed, even in the absence of detailed distributional knowledge, generates at least 97% of the surplus an optimal mechanism would produce.

## 6 Extensions

While the main result is quite general with respect to the handicapping rule \( \theta \), it is of both theoretical and practical interest to test its robustness with respect to the distributional assumptions. The main stumbling block in extending Theorem 4 is the construction in Theorem 3 which is where the restriction on the shape of \( f \) originates. The expression in (10) and ,by implication, the \( \rho \)-concavity of the distribution, is thus at the core of any potential argument extending our ranking beyond the class of increasing densities. This link is the key by which the analysis can be extended.

First, observe that every term involved in (10) is positive, which is clearly, too strong a requirement. As soon as we move beyond the class of increasing densities Lemma 2 implies that the first term becomes negative.\(^{24}\) The key is to make the transition in such a way as to maintain the overall sign of the expression. The proof of the next result exploits key properties of \( \rho \)-concavity to provide a constructive method to achieve this.\(^{25}\)

Consider a distribution \( F \) with an increasing density \( f \). Construct a new distribution by considering \( G = F^n \), for some \( n > 1 \). If \( n \) is an integer \( G \) is easily interpretable as the distribution of the minimum of \( n \) independent draws out of \( F \). At the heart of the next result is the observation that if

\(^{24}\) Log-concavity of \( f \) implies that once \( f' \) is negative it will stay negative. Thus \( W_{f,F} > \frac{1}{2} \) over that range.

\(^{25}\) In the process we prove some novel results on \( \rho \)-concavity.
the expression (10) associated with $f$ is positive the analogous expression associated with $g$ is also positive.\textsuperscript{26} Thus we can extend Theorem 4.

**Theorem 5** If $f$ is increasing and $W_F$ is decreasing then for any $n > 1$ consider $g = n f F^{n-1}$. Then, if bidders’ costs are distributed according to $g$ any FP-HR\textsubscript{a} yields an ex-post inferior allocation to that of an appropriately chosen SPBA.

Note that in the newly constructed class of distributions we have $g(1) = 0$. Thus it includes strictly uni-modal or even decreasing densities.\textsuperscript{27}

Mares and Swinkels (2010a,2010c), in their analysis of FPHAs, extend this type of reasoning to situations when costs are asymmetric and when we have more than 2 players. Each of these situations require different techniques and distributional assumptions, which have no immediate counterpart in FP-HRs, but show that the main insight behind the ranking results is robust.

## 7 Conclusion

The main object of this paper is to understand the implications of prevalent procurement mechanisms which rely on first-price designs and handicapping rules. While of great practical importance these mechanisms are challenging to a theorist since at their core they have an asymmetric first price auction. They are also challenging to practitioners. Buyers and bidders face a hard problem when designing respectively participating under these rules. While the handicapping skews in equilibrium the allocation towards the optimum, it is also obvious that it does so in an uneven and sometimes counterproductive way. The question of finding the optimal percentage handicap is daunting even under the simplest of distributional assumptions since its equilibrium effects on bidding are not transparent. To put it mildly, a designer of proportional handicap rules will have a very hard time arguing why he picked 15% instead of 25% or 5%.

On the seller side, there are at least two practical questions raised by FP-HRs aside from the obvious equilibrium bid formulation problem. For non-preferred bidders at the interim stage, the question of participation becomes central. More surprisingly, maybe is that preferred bidders who are better off ex-ante also face asymmetric incentives at the interim stage. In particular,

\textsuperscript{26}This is another significant methodological departure from Mares and Swinkels (2010a).

\textsuperscript{27}The latter is true if $n$ is large enough and $f(0) > 0$. In this case the requirement on concave virtual costs is satisfied automatically since $W_G$ is decreasing when $g$ is decreasing.
regardless of the form of the handicap, preferred efficient suppliers are chosen less often than they should under optimal rules. At the other end, inefficient preferred suppliers tend to enjoy extra rents relative to the optimum, which blurs participation incentives again. Overall, while such mechanism can generate modest cost savings (see Cabral and Greenstein (1990), Corns and Schotter (1999)) over standard auctions, our main argument shows that SPBAs, as simple and practical to implement, will do uniformly better.

This paper leaves a key unanswered questions for future research. If SPBAs are ex-post better, why are firms and agencies persisting in their commitment to FP-HRs? Mares and Swinkels (2011) explore a potential answer, by endogenizing the buyer’s preferences. In this paper the preferences of the buyer are fixed, but if one or more of the bidders can unilaterally make costly, observable investments in the “quality” of their product then the handicapping rules also become ways by which a seller can offer incentives for such actions. Suboptimal and dominated mechanisms become more attractive in this light. They might offer lower surpluses for fixed quality levels, but might compensate the buyer by inducing higher investments in quality.\textsuperscript{28}

\textsuperscript{28}See Asker and Cantillon (2006) for a somewhat similar approach.
References


[37] Rezende, Leonardo (2004); *Biased Procurement Auctions*, working paper, University of Illinois.


[40] Shachat, Jason and Swarthout, Todd J. (2003); *Procurement Auctions for Differentiated Goods*, working paper IDEAS Experimental 0310004.


8 Appendix

Proof of Theorem 1 If 1 with type $c$ bids as if his type is $\hat{c}$, his surplus is

$$S_1'(c) = \frac{\partial}{\partial \hat{c}} \hat{S}_1(\hat{c}; c) \bigg|_{\hat{c}=c} = -\bar{F}(\psi(c)).$$

(11)

Given that $b_1$ is restricted to be at most 1, $S_1(1) = 0$, yielding (4). Similarly,

$$S_2'(c) = -\bar{F}(\phi(c)),$$

(12)

and for $c_2 > \bar{A}$ no $b_2 > c_2$ ever wins, and so $S_2(\bar{A}) = 0$, yielding (5).

Since $\bar{F}(\psi(c))(\beta_1(c) - c) = S_1(c)$, and by (11), $\beta_1(c) = c + \frac{S_1(c)}{F_1(c)}$.

But then, wherever $\psi$ is differentiable,

$$\beta_1(c) = 1 + \frac{S_1'(c) (-S_1'(c) + S_1(c) S_1''(c))}{(S_1'(c))^2} = W_{S_1}(c),$$

giving (6), and similarly for (7).

As strictly increasing functions, $\beta_1$ and $\beta_2$ are differentiable almost everywhere. And, as $\beta_1(\phi(c)) = \theta(\beta_2(\phi(c)))$, $\phi$ is continuous, increasing and thus almost everywhere differentiable, and where $\beta_1'$ and $\beta_2'$ exist,

$$\phi'(c) = \theta'(\beta_2(c)) \frac{\beta_2'(c)}{\beta_1'(\phi(c))} > 0.$$  

(13)

Substituting (6) and (7) into (13) gives

$$\phi'(c) = \theta'(\beta_2(c)) \frac{W_{S_2}(c)}{W_{S_1}(\phi(c))} = \theta'(\beta_2(c)) \frac{S_2(c)}{S_1(\phi(c))} \frac{f(\phi(c)) \phi'(c)}{F_2(\phi(c)) \phi'(c)},$$

(14)

using (11) and (12) to evaluate $W_{S_2}$ and $W_{S_1}$. Canceling $\phi'(c) > 0$ and rearranging yields (8).

Now, let us show that $\phi \in C^{k+1}[0, \bar{A}]$. We show the result on $[0, a]$, $a < \bar{A}$. Since $a$ is arbitrary, the result follows. We have that $\phi$ is $C^0[0, a]$.

Assume that $\phi \in C^k[0, a]$ where $0 \leq \hat{k} \leq k$. Then, $f(\phi)$ and $\bar{F}(\phi)$ belong to $C^k[0, a]$. Since $S_2(c) = \int_c^A \bar{F}(\phi(s))ds$ it follows from the fundamental theorem of calculus that $S_2 \in C^{k+1}[0, a]$ and similarly for $S_1$. But, as a bounded,
continuous function on a compact interval, \( \phi \) is absolutely continuous and so
\[
\phi(c) = \phi(0) + \int_0^c \phi'(t) \, dt = \int_0^c \frac{1}{\theta'(\beta_2(c))} \frac{S_1(\phi(t))}{S_2(t)} \frac{f(c)}{F_2(c)} \frac{f(\phi(t))}{F_2(\phi(t))} \, dt.
\]
As each part of the integrand is positive and belongs to \( C^k[0,a] \), it follows that \( \phi \in C^{k+1}[0,a] \). By induction, \( \phi \in C^{k+1}[0,1] \) and \( \beta_2 \in C^{k+1}[0,A] \).

**Proof of Theorem 3** From Theorem 1

\[
\phi'(r) = \frac{1}{\theta'(\beta_2(r))} \frac{S_1(\phi(r))}{S_2(r)} \frac{f(r)}{F_2(c)} \frac{f(\phi(r))}{F_2(\phi(r))}.
\]
Denote
\[
S_1(\phi(c)) \frac{f(c)}{F_2(c)} \equiv T \quad \text{and} \quad S_2(r) \frac{f(\phi(r))}{F_2(\phi(r))} \equiv B.
\]
Then \( \frac{\phi''}{\phi'} = (\log \phi')' \) is
\[
-\frac{\theta''(\beta_2(r)) \beta_2'(r)}{(\theta'(\beta_2(r)))^2} + \phi'(r) \frac{S_1'(\phi(r))}{S_1(r)} - \frac{S_2'(\phi(r))}{S_2(r)} + \left( \log \frac{f}{F_2(r)} \right)'(r) - \phi'(r) \left( \log \frac{f}{F_2(r)} \right)'(\phi(r)),
\]
and so by the concavity of \( \theta \) we conclude that
\[
\frac{\phi''}{\phi'}(r) \geq \phi'(r) \frac{S_1'(\phi(r))}{S_1(r)} - \frac{S_2'(\phi(r))}{S_2(r)} + \left( \log \frac{f}{F_2(r)} \right)'(r) - \phi'(r) \left( \log \frac{f}{F_2(r)} \right)'(\phi(r)).
\]
By log-concavity of \( \tilde{F} \), \( \left( \log \frac{f}{F_2(r)} \right)' > 0 \), and so, since \( \phi'(r) \leq 1 \) by assumption
\[
\left( \log \frac{f}{F_2(r)} \right)'(r) \geq \phi'(r) \left( \log \frac{f}{F_2(r)} \right)'(r).
\]
Note that \( \left( \log \frac{f}{F_2(r)} \right)'(r) = \frac{f'(r)}{f(r)} - 2 \frac{\tilde{F}}{F_2(r)}(r) \). From (4), \( S_1'(\phi(r)) = -\tilde{F}(r) \), and so
\[
\frac{S_1'(\phi(r))}{S_1(\phi(r))} = -\frac{\tilde{F}(r)}{T}.
\]
and similarly
\[
\frac{S_2'(r)}{S_2(r)} = -\frac{\tilde{F}(r)}{B} = -\phi'(r) \frac{\tilde{F}(\phi(r))}{T}.
\]
Substitute (17), (18), and (19) into (16), collect terms, and cancel \( \phi'(r) > 0 \) to obtain
\[
\phi''(r) \geq \left( \frac{1}{T} - 2 \right) \left( \frac{f}{F}(\phi(r)) - \frac{f}{F}(r) \right) + \left( \frac{f'}{f}(r) - \frac{f'}{f}(\phi(r)) \right).
\]  
(20)

Now,
\[
S_2(r) = \int_r^{\phi(r)} \frac{F(s)ds}{\phi(r)} - \int_0^{\phi(r)} \frac{F(s)ds}{\phi(r)} < \frac{1}{\phi'(r)} \int_0^1 F(s)ds,
\]
where the strict inequality follows since \( r \) is a global minimum of \( \phi' \), with \( \phi'(r) \leq 1 \), since \( \phi' \) is continuous, and since \( \lim_{\varepsilon \to 0} \varepsilon \phi'(\varepsilon) > 1 \). Multiplying both sides by \( \phi'(r) \) yields \( \phi'(r) B < W_f \frac{F(\phi(r))}{\phi'(r)} \). Since \( T = \phi'(r) B \), we have
\[
T < W_f \frac{F(\phi(r))}{\phi'(r)}.
\]  
(21)

Substituting (21) into (20) yields the result. ■

**Proof of Lemma 2** If \( f \) is increasing then \( F \) is concave and thus at every point \( c \) the graph \( F \) lies below its tangent of slope \( -f(c) \). Thus \( \int_c^F F(s)ds \) the area under the graph is bounded by the triangle of height \( F(c) \) which lies below the tangent and above the axis. This triangle has an area of \( \frac{F^2(c)}{2f(c)} \) and thus we have
\[
\int_c^F F(s)ds \leq \frac{F^2(c)}{2f(c)}
\]
or \( W_f \leq \frac{1}{2} \). The inequalities are strict if \( f \) is strictly increasing. The proof for the decreasing case is analogous. ■

For the proof of Lemma 3 we will need to establish a preliminary technical aid first.

**Lemma 5** Consider \( f : [0, 1) \to R \) is continuously differentiable with \( f(0) = 0 \). If for all \( x > 0 \), \( f(x) \leq 0 \) implies \( f'(x) > 0 \) then \( f(x) \geq 0 \). Further, if \( f(x') = 0 \) for some \( x' > 0 \) then \( f(x) = 0 \) for all \( x \in [0, x'] \).

**Proof of Lemma 5** If \( f(x_0) > 0 \) for some \( x_0 > 0 \) then \( f(x) > 0 \) for all \( x \in [x_0, 1) \). To see this note that the function cannot cross zero from above
since at every point where \( f(x) = 0 \) we have \( f'(x) > 0 \). But then imagine there exists some \( x_1 > 0 \) such that \( f(x_1) < 0 \). Then there exists \( x_2 \in (0, x_1) \) such that \( f'(x_2) < 0 \) and thus \( f(x_2) > 0 \), a contradiction. The last statement follows easily since for any \( x' > 0 \) with \( f(x') = 0 \), \( f(x) \) for \( x < x' \) can not be strictly positive or negative.

**Proof of Lemma 3** From Theorem 1, for any \( c_2 > 0 \) such that \( c_1 = \phi (c_2) \leq c_2 \) we have

\[
\phi'(c_2) = \frac{1}{\theta'(\beta_2(c_2))} \left( \frac{\beta_1(c_1) - c_1}{\beta_2(c_2) - c_2} \right) \left( \frac{\theta(\beta_2(c_2)) - \theta(c_2)}{\beta_2(c_2) - c_2} \right) \quad \text{for all } c_2 > 0.
\]

The bracketed term in (22) equals, by the intermediate value theorem, \( \theta'(\xi) \). But since \( \theta' \) is decreasing it follows that the ratio \( \frac{\theta'(\xi)}{\theta'(\beta_2(c_2))} \) is at least 1. Thus \( \phi(c) < c \) implies \( \phi'(c) > 1 \) so that \( \phi' \) crosses \( c \) at most once and from below on \((0, A)\), which using Lemma 5 concludes the proof of (A).

In order to see (B) proceed by a similar argument as before, pick \( c_1 = \phi(c_2) = \theta(c_2) > c_2 \) then from Theorem 1

\[
\phi'(c_2) = \frac{1}{\theta'(\beta_2(c_2))} \left( \frac{\beta_1(c_1) - c_1}{\beta_2(c_2) - c_2} \right) \left( \frac{\theta(\beta_2(c_2)) - \theta(c_2)}{\beta_2(c_2) - c_2} \right) \quad \text{for all } c_2 > 0.
\]

Note that since \( \theta \) is concave with \( \theta' \geq 1 \)

\[
\frac{1}{\theta'(\beta_2(c_2))} \left( \frac{\theta(\beta_2(c_2)) - \theta(c_2)}{\beta_2(c_2) - c_2} \right) \leq \left( \frac{\theta(\beta_2(c_2)) - \theta(c_2)}{\beta_2(c_2) - c_2} \right) \leq \theta'(c_2)
\]

and that since \( \theta(c) > c \) for all \( c > 0 \) the last term in (23) is strictly less than 1, which yields that \( \phi'(c_2) < \theta'(c_2) \) whenever \( \phi(c_2) = \theta(c_2) \) concluding the proof of (B).
Proof of Lemma 4 For part (A) apply the same argument as in the proof of Lemma 4 to the case \( \phi(0) = 0 \) which yields \( \phi'(0) \geq 1 \). Further, if \( \theta(0) > 0 \) or if \( \theta \) is concave then since \( \beta_2(0) > 0 \) we \( \frac{\theta(\beta_2(0))}{\theta'(\beta_2(0))\beta_2(0)} > 1 \) so we can conclude that \( \phi'(0) > 1 \), yielding the desired result.

For part (B) we present a proof based on the assumption that each of \( \beta_1', \beta_2 \) and \( \phi' \) has a well defined limit in the extended real line as \( c \to A \). A (surprisingly involved) complete proof is available in Mares and Swinkels (2008). First, let us show that \( \beta_1(1) = \beta_2(A) = 0 \). To see this, note first that for any \( c < 1 \), \( \beta_1(c) \) earns \( F(\psi(c)) (\beta_1(c) - c) \), while a bid of 1 earns at least \( F(A) (1 - c) \), and so

\[
F(\psi(c)) (\beta_1(c) - c) \geq F(A) (1 - c),
\]

from which

\[
\frac{\beta_1(c) - c}{1 - c} \geq \frac{F(A)}{F(\psi(c))},
\]

and so, since \( \psi(1) = A \),

\[
\lim_{c \to 1} \frac{\beta_1(c) - c}{1 - c} \geq 1. \tag{24}
\]

But, by definition,

\[
\lim_{c \to 1} \frac{\beta_1(1) - \beta_1(c)}{1 - c} = \beta_1'(\bar{c}_1). \tag{25}
\]

Adding (24) and (25),

\[
\lim_{c \to 1} \frac{\beta_1(1) - c}{1 - c} \geq 1 + \beta_1'(1).
\]

Since \( \beta_1(1) = 1 \), it follows that \( \beta_1'(1) = 0 \).

The first order condition for player 1’s profit at \( c < 1 \) is

\[
F(\psi(c)) \beta_1'(c) = f(\psi(c)) \psi'(c) (\beta_1(c) - c).
\]

But, \( \psi'(c) = \frac{\beta_1'(c)}{\theta'(\beta_2(\psi(c)))\beta_2(\psi(c))} \), and so

\[
F(\psi(c)) \beta_1'(c) = f(\psi(c)) \frac{\beta_1'(c)}{\theta'(\beta_2(\psi(c)))\beta_2(\psi(c))} (\beta_1(c) - c).
\]

Cancelling \( \beta_1'(c) > 0 \), and rearranging,

\[
0 < \beta_2'(\psi(c)) = \frac{f(\psi(c))}{\theta'(\beta_2(\psi(c))) F(\psi(c))} (\beta_1(c) - c) < \frac{1}{\theta'(\beta_2(\psi(c))) F(\psi(c))} f(A) (1 - c).
\]
Thus, as $c \to 1$, $\beta_2' (\psi (c)) \to 0$, i.e., $\beta_2' (\bar{A}) = 0$.

Now, let us show that $\lim_{c \to \bar{A}} \phi' (c) \notin (0, \infty)$. To see this, note that

$$
\lim_{c \to \bar{A}} \frac{\beta_1 (\phi (c)) - \phi (c)}{\beta_2 (c) - c} = \lim_{c \to \bar{A}} \frac{\beta_1' (\phi (c)) - 1}{\beta_2' (c) - 1} \phi' (c) = \lim_{c \to \bar{A}} \phi' (c),
$$

by l'Hôpital's rule and the previous step. Assume that $\lim_{c \to \bar{A}} \phi' (c) = \alpha \in (0, \infty)$. By Theorem 1

$$
\phi' (c) = \frac{1}{\theta' (\beta_2 (c))} \frac{\beta_1 (\phi (c)) - \phi (c)}{\beta_2 (c) - c} \frac{f(c)}{F(c)},
$$

and so by (26), we have

$$
\alpha = \alpha \lim_{c \to \bar{A}} \frac{1}{\theta' (\beta_2 (c))} \frac{f(c)}{F(c)} = 0,
$$

since $\phi (c) \to 1$ and $\theta' > 0$, while $\bar{A} < 1$, a contradiction.

So, assume that $\lim_{c \to \bar{A}} \phi' (c) = 0$. Then, for any small $t$, there is a last $c (t)$ at which $\phi' (c) = t$ (this is well defined since $\phi$ is continuously differentiable and $[0, \bar{A}]$ is compact). But, by a change of variables,

$$
S_1 \left( \phi (c (t)) \right) = \int_{\phi (c (t))}^{1} \bar{F} (\psi (s)) \, ds
$$

$$
= \int_{c (t)}^{\bar{A}} \bar{F} (s) \phi' (s) \, ds,
$$

$$
< t (\bar{A} - c (t)),
$$

since $\phi' (s) < t$, and $\bar{F} < 1$ for $s > c (t)$.

Thus, since $\frac{f}{\bar{F}}$ is increasing

$$
S_1 \left( \phi (c (t)) \right) \frac{f}{\bar{F}^2} (c (t)) < t (\bar{A} - c (t)) \frac{f}{\bar{F}^2} (\bar{A}).
$$

The RHS converges to 0 as $t \to 0$ and $c (t) \to \bar{A}$. But then the term $\frac{1}{t} - 2$ in (20) diverges for $r = c (t)$ for small $t$, the term $\frac{f(\phi (c (t)))}{f(\phi (c (t)))} - \frac{f(c (t))}{f(c (t))}$ diverges as well (noting that $c (t) < 1 - \bar{A}$), and, by log-concavity of $f$, the remaining term does not go to $-\infty$. Hence, $\phi'' (c (t)) > 0$, contradicting the construction of $c (t)$.$^{29}$

$^{29}$At the point that (20) is derived, we have used only that $\phi' (r) < 1$, which holds for $r = c (t)$ for small $t$ by definition, and none of the other properties assumed in the statement of Theorem 3.
Since we have ruled out \( \lim_{c \to \Delta} \phi'(c) \in [0, \infty) \), we have that \( \lim_{c \to \Delta} \phi'(c) = \infty \). □

**Proof of Theorem 5** We will proceed in several steps.

**Step 1**: If \( f \) is increasing then by strict log-concavity \( f' \) and \( \frac{f}{c} \) are positive, strictly decreasing functions and thus \( W_{f} \) is strictly decreasing. Further \( W_{f}(c) > 0 \) and \( W_{f}(c) > 0 \) for all \( c \in [0, 1) \).

**Step 2**: If \( f \) is increasing then \( W_{f} \) is strictly increasing for all \( c < 1 \).

To see this note by l’Hopital’s rule that

\[
W_{f}(1) = \lim_{c \to 1} f(c) \frac{\int_{c} F(s) ds}{F^{2}(c)} = f(1) \lim_{c \to 1} \frac{f(c)}{F(c)} = \frac{1}{2},
\]

while \( W_{f}(1) = 0 \).

Define

\[
K(c) = -\frac{1}{W_{f} F(c)} + W_{F}(c) + 2,
\]

and observe that

\[
\frac{\partial}{\partial c} \ln W_{f} F(c) = -\frac{F(c)}{f(c)} + f'(c) + \frac{f(c)}{F(c)} = \frac{f(c)}{F(c)} K(c).
\]

Note also that \( \lim_{c \to 1} K(c) = 0 \) and that

\[
\frac{\partial}{\partial c} K(c) = \left(\frac{W_{f} F(c)}{W_{F}(c)}\right)' = \frac{W_{f}'(c)}{W_{f}(c)} + W_{f}'(c).
\]

But then if \( K(c) \leq 0 \) at some \( c \) we have that \( \left(\frac{W_{f} F(c)}{W_{F}(c)}\right)' < 0 \) and thus \( \frac{\partial}{\partial c} K(c) < 0 \). Since \( K(1) = 0 \) we have that this is an impossibility, and thus \( K(c) > 0 \) for all \( c < 1 \).

**Step 3**: If \( f \) is increasing then \( W_{f} F_{n} \) is strictly increasing for all \( n > 1 \) and for all \( c < 1 \).

Start by noting that since \( W_{f} F_{n}(c) = \frac{n f(c) \int_{c} F^{n}(s) ds}{F^{n+1}(c)} \) and thus by l’Hopital’s rule we have

\[
W_{f} F_{n}(1) = \lim_{c \to 1} \frac{n f(c) \int_{c} F^{n}(s) ds}{F^{n+1}(c)} = n f(1) \lim_{c \to 1} \frac{F^{n}(c)}{f(c) F^{n}(c)} = \frac{n}{n + 1}.
\]
By analogy with Step 2 define

$$K_n(c) \equiv -\frac{n}{W_{f}^{n}}(c) + W_{F}(c) + (n + 1)$$

and observe that \( \lim_{c \to 1} K_n(1) = 0 \) and

$$\frac{\partial}{\partial c} \ln W_{f}^{n}(c) = -\frac{F^n(c)}{\int_{c}^{1} F^n(s) ds} + \frac{f'(c)}{f(c)} + (n + 1) \frac{f(c)}{F(c)}$$

$$= \frac{f(c)}{F(c)} K_n(c).$$

Note also that

$$\frac{\partial}{\partial c} K_n(c) = n \left( \frac{W_{f}^{n}(c)}{W_{f}^{n}(c)} \right)' + W_{F}'(c).$$

Similarly as in Step 2 observe that if \( K_n(c) \leq 0 \) for some \( c \in [0, 1) \) we have that \( \left( W_{f}^{n}(c) \right)' \leq 0 \) and thus \( \frac{\partial}{\partial c} K_n(c) < 0 \), which constitutes a contradiction to \( \lim_{c \to 1} K_n(1) = 0 \). This implies that \( K_n(c) > 0 \) for all \( c \in [0, 1) \).

**Step 4:** If \( f \) is increasing then \( \frac{W_{f}^{n}}{W_{f}^{n}} \) is increasing for all \( n > 1 \) and all \( c \in [0, 1) \).

Using the derivations in Steps 2 and 3 observe that

$$\frac{\partial}{\partial c} \ln \frac{W_{f}^{n}}{W_{f}^{n}}(c) = \frac{\partial}{\partial c} \ln W_{f}^{n}(c) - \frac{\partial}{\partial c} \ln W_{F}(c)$$

$$= \frac{f(c)}{F(c)} \left( K_n(c) - K(c) \right)$$

$$= \frac{f(c)}{F(c)} \left( -\frac{n}{W_{f}^{n}}(c) + \frac{1}{W_{f}^{n}}(c) + (n - 1) \right)$$

$$= \frac{f(c)}{F(c) W_{f}^{n}(c)} \left( -n \frac{W_{f}^{n}}{W_{f}^{n}}(c) + 1 + (n - 1) W_{f}^{n}(c) \right).$$

Define

$$K_{1,n}(c) = -n \frac{1}{W_{f}^{n}}(c) + 1 + (n - 1) W_{f}^{n}(c)$$
and observe by steps 2 and 3 that $\lim_{c \to 1} K_{1,n}(1) = 0$

$$\frac{\partial}{\partial c} K_{1,n}(c) = \frac{n \left( \frac{W_{f, F_n}(c)}{W_{f, \bar{F}}(c)} \right)'}{\left( \frac{W_{f, F_n}(c)}{W_{f, \bar{F}}(c)} \right)^2} + (n - 1) W'_{f, \bar{F}}(c).$$

Thus if $K_{1,n}(c) \geq 0$ for some $c < 1$ we have $\left( \frac{W_{f, F_n}(c)}{W_{f, \bar{F}}(c)} \right)' \geq 0$ and since $W'_{f, \bar{F}}(c) > 0$ we conclude that $\frac{\partial}{\partial c} K_{1,n}(c) > 0$. This is a contradiction to $\lim_{c \to 1} K_{1,n}(1) = 0$ and thus $\left( \frac{W_{f, F_n}(c)}{W_{f, \bar{F}}(c)} \right)' < 0$ for all $c < 1$. We conclude by observing that therefore

$$\frac{n}{W_{f, \bar{F}}(c)} \geq \frac{1}{W_{f, \bar{F}}(c)} + (n - 1)$$

with strict inequality whenever $c < 1$.

Step 5: Consider $\bar{G} = \bar{F}^n$ for some $n > 1$, then $g = n f \bar{F}^{n-1}$ and $g' = n f' \bar{F}^{n-1} - n(n - 1) f^2 \bar{F}^{n-2}$. Recall that by Theorem 3 a sufficient condition $\phi' > 1$ is that at an interior global minimum where $\phi'(r) < 1$ we have $H(\phi'(r)) > 0$. In turn this condition is equivalent to

$$\frac{1}{W_{f, \bar{F}}(\phi'(r))} > 2 + \frac{f'(\phi(r)) - f'(r)}{f(\phi(r)) - f(r)} \equiv X_1$$

$$\equiv Y_1$$

which is satisfied for all increasing $f$ and $\phi'(r) \geq r$ by Lemma 2.\footnote{The exact location of $r$ and $\phi(r)$ is irrelevant for this proof. The only thing that matters is that $\phi(r) \geq r$.}

We want to show that when $f$ is increasing (27) implies

$$\frac{1}{W_{f, \bar{G}}(\phi(r))} > 2 + \frac{g'(\phi(r)) - g'(r)}{g(\phi(r)) - g(r)} \equiv X_2$$

$$\equiv Y_2$$

for all $\phi(r) \geq r$. Observe that

$$\frac{g'}{g} = \frac{f'}{f} - (n - 1) \frac{f}{\bar{F}}$$
and that
\[ \frac{g}{G} = \frac{n f}{F}. \]

Combining the two results yields
\[ Y_2 = \frac{1}{n} Y_1 - \frac{n - 1}{n}, \]

but by (27) \( X_1 - 2 > Y_1 \) so a sufficient condition for (28) is that
\[ X_2 - 2 \geq \frac{1}{n} (X_1 - 2) - \frac{n - 1}{n} = \frac{1}{n} (X_1 - 1) - 1, \]
or equivalently
\[ nX_2 \geq X_1 + (n - 1). \]  \hspace{1cm} (29)

Substituting back for \( X_1 \) and \( X_2 \) we have that (29) is equivalent to
\[ \frac{n}{W_{fG}(\phi(r))} \geq \frac{1}{W_{fF}(\phi(r))} + (n - 1) \]

which is true by Step 4.

Since the proof of Lemmas 3 and 4 do not depend of \( f \) being increasing we need only to establish a contradiction to Theorem 3. We conclude by stating that if \( f \) is increasing and \( G = F^n \) then for all \( 1 > \phi(r) \geq r \)
\[ \left( \frac{1}{W_{fG}(\phi(r))} - 2 \right) \left( \frac{g}{G}(\phi(r)) - \frac{g}{G}(r) \right) + \frac{g'}{g}(r) - \frac{g'}{g}(\phi(r)) > 0 \]

and in particular at any interior global minimum \( r \) with \( \phi'(r) < 1 \). This is a contradiction to Theorem 3 and thus proves by Lemmas 3 and 4 that \( \phi' \geq 1 \). \( \blacksquare \)